LINEAR SYSTEMS, MATRICES, AND VECTORS

Now that I've been teaching Linear Algebra for a few years, I thought it would be great to integrate the more advanced topics such as vector spaces, the Euclidean dot product, and matrix operations early on in our class, instead of hurrying to fit everything in late in the course. So…hold on to your seats...we're in for a bumpy ride!

1.1 Linear Systems and Matrices

Learning Objectives

1. Use back-substitution and Gaussian elimination to solve a system of linear equations
2. Determine whether a system of linear equations is consistent or inconsistent
3. Find a parametric representation of a solution set
4. Write an augmented or coefficient matrix from a system of linear equations
5. Determine the size of a matrix

Let's Do Our Math Stretches!

1. Solve the following systems of linear equations

a.

$$\begin{aligned} -x+8y &= 3 \quad R1 \\ 6x &= 12 \quad R2 \end{aligned}$$

$$\begin{aligned} -x+8y&=3 \\ 6x+0y&=12 \end{aligned} \rightarrow \tfrac{1}{6}R2 \rightarrow \begin{aligned} -x+8y&=3 \\ x&=2 \end{aligned} \rightarrow y=\tfrac{5}{8}$$

$\boxed{\{(2,\tfrac{5}{8})\}}$

consistent system with independent equations

b.

$$\begin{aligned} 3x+y-z&=15 \rightarrow x=6 \\ 2y+4z&=0 \rightarrow y=-2 \\ z&=1 \end{aligned}$$

$\{(6,-2,1)\}$

consistent system w/ independent equations

DEFINITION OF A LINEAR EQUATION IN *n* VARIABLES

A linear equation in *n* variables $x_1, x_2, x_3, \ldots, x_n$ has the form

$$a_1x_1 + a_2x_2 + a_3x_3 + \cdots + a_nx_n = b$$

The coefficients $a_1, a_2, a_3, \ldots, a_n$ are real numbers, and the constant term b is a real number. The number a_1 is the leading coefficient, and x_1 is the leading variable.

*Linear equations have no products or roots of variables and no variables involved in transcendental functions.

Example 1: Give an example of a linear equation in three variables.

$$4x_1 + 0x_2 + 3x_3 = 10 \rightarrow 4x_1 + 3x_3 = 10$$

DEFINITION OF SOLUTIONS AND SOLUTION SETS

A solution of a linear equation in *n* variables is a sequence of *n* real numbers $s_1, s_2, s_3, \ldots, s_n$ arranged to satisfy the equation when you substitute the values

$$x_1 = s_1, x_2 = s_2, x_3 = s_3, \ldots, x_n = s_n$$

into the equation. The set of all solutions of a linear equation is called its solution set, and when you have found this set, you have satisfied the equation. To describe the entire solution set of a linear equation, use a parametric representation.

Example 2: Solve the linear equation.

$x_1 + x_2 = 10 \rightarrow x_1 = 10 - x_2$

Let $x_2 = t$, $x_1 = 10 - t$

such that

belongs to / is an element of

$$\{\underbrace{(10-t, t)}_{\text{describe the form of the solution(s)}} : \underbrace{t \in \mathbb{R}}_{\text{explanation of the format of any parameters and/or limitations}}\}$$

Example 3: Solve the linear equation.

$2x_1 - x_2 + 5x_3 = -1.$

$x_1 = \frac{1}{2}(x_2 - 5x_3 - 1)$

let $x_2 = t$, $x_3 = s$

$x_1 = \frac{1}{2}(t - 5s - 1)$

$$\boxed{\left\{ \left(\frac{1}{2}(t-5s-1), t, s \right) : s, t \in \mathbb{R} \right\}}$$

SYSTEMS OF LINEAR EQUATIONS IN *n* VARIABLES

A system of linear equations in *n* variables is a set of *m* equations, each of which is linear in the same *n* variables.

$$\begin{array}{ccccccccc}
a_{11}x_1 & + & a_{12}x_2 & + & a_{13}x_3 & + \cdots + & a_{1n}x_n & = & b_1 \\
a_{21}x_1 & + & a_{22}x_2 & + & a_{23}x_3 & + \cdots + & a_{2n}x_n & = & b_2 \\
a_{31}x_1 & + & a_{32}x_2 & + & a_{33}x_3 & + \cdots + & a_{3n}x_n & = & b_3 \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
a_{m1}x_1 & + & a_{m2}x_2 & + & a_{m3}x_3 & + \cdots + & a_{mn}x_n & = & b_m
\end{array}$$

SOLUTIONS OF SYSTEMS OF LINEAR EQUATIONS

A solution of a system of linear equations is a __Sequence__ of numbers $s_1, s_2, s_3, \ldots, s_n$ that is a solution of each of the linear equations in the __System__.

Example 4: Graph the following linear systems and determine the solution(s), if a solution exists.

a.

$x - y = 8$

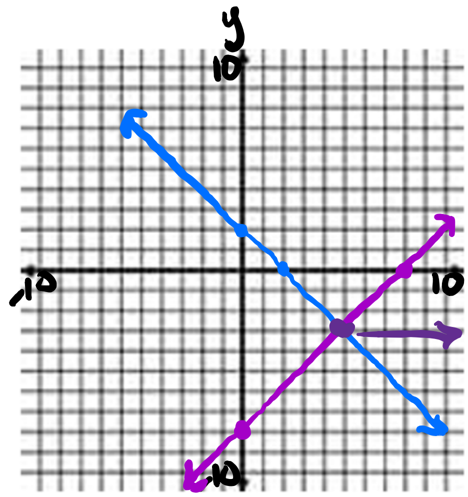
$x + y = 2$

$2x = 10$

$x = 5$

$y = -3$

$\{(5,-3)\}$
consistent system
w/ independ. equations



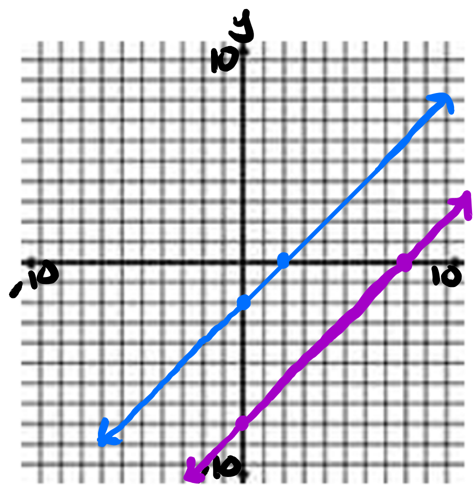
(5,-3)

b.

$x - y = 8$

$x - y = 2$

$\{\ \}$ or $\emptyset$
inconsistent system
with independent
equations



parallel
lines
no
solution

c.

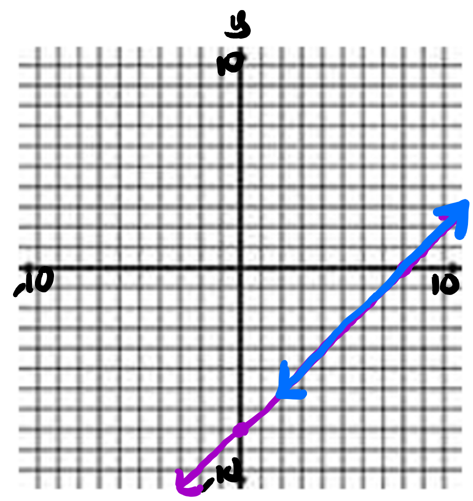
$2x - 2y = 16 \rightarrow x - y = 8$

$3x - 3y = 24 \rightarrow x - y = 8$

$x - y = 8 \rightarrow x = y + 8 \rightarrow x = t + 8$

Let $y = t$

$\{(t+8, t) : t \in R\}$
consistent system
with dependent equations



collinear
infinitely
many
solutions

NUMBER OF SOLUTIONS OF A SYSTEM OF EQUATIONS

For a system of linear equations, precisely one of the following is true.

The system has exactly one solution. (consistent system).

The system has infinitely many solutions (consistent system)

The system has no solution (inconsistent system).

TYPES OF SOLUTIONS

2 Equations, 2 Variables

What did we learn from the last example?

Inconsistent:

parallel lines

Consistent:

cross at one point
or collinear

3 Equations, 3 Variables

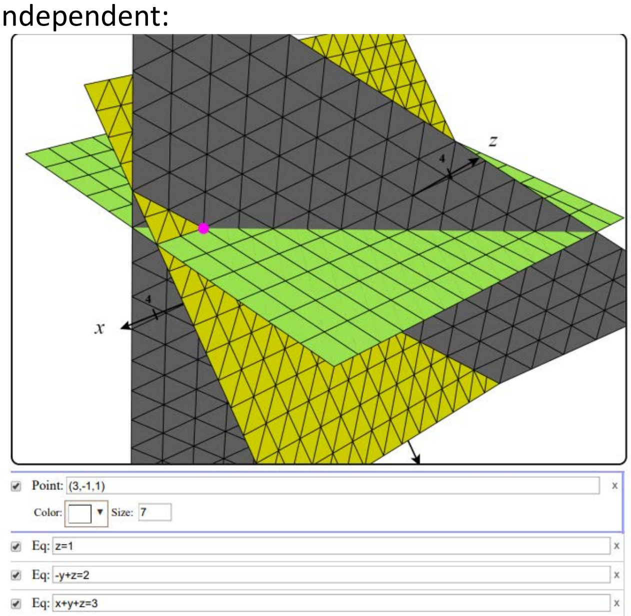
Inconsistent

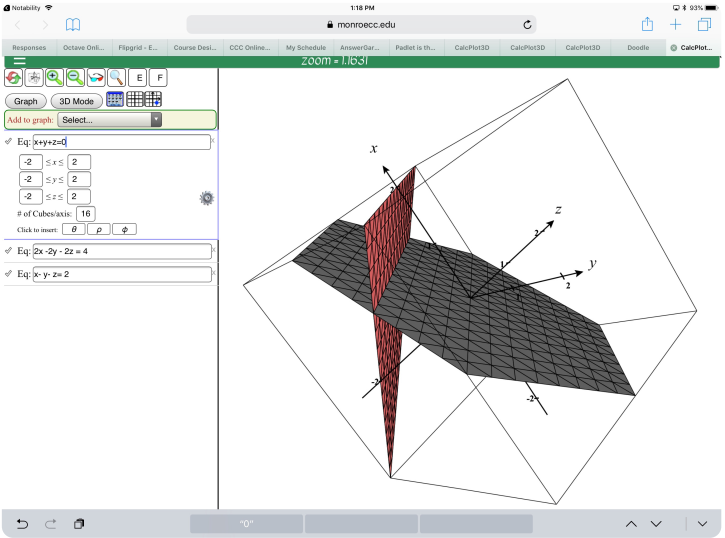
Parallel Planes Intersecting Two at a Time (1) or Intersecting Two at a Time (2)

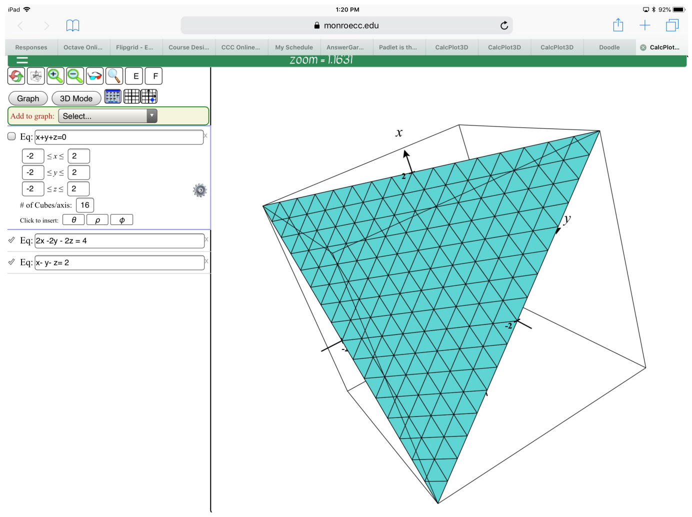
Consistent

Dependent: Linear Intersection Planar Intersection

Independent:




monroecc.edu
zoom = 1.1631
Graph
3D Mode
Add to graph:
Select...
Eq: x+y+z=0
# of Cubes/axis: 16
Click to insert:
Eq: 2x -2y - 2z = 4
Eq: x- y- z= 2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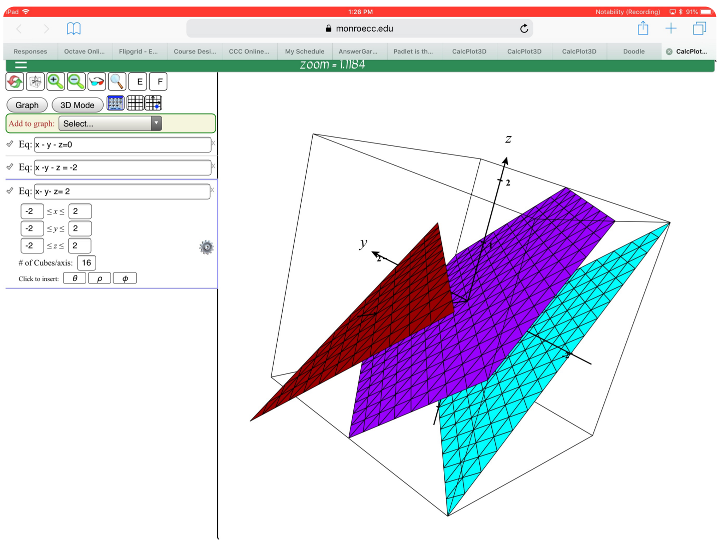
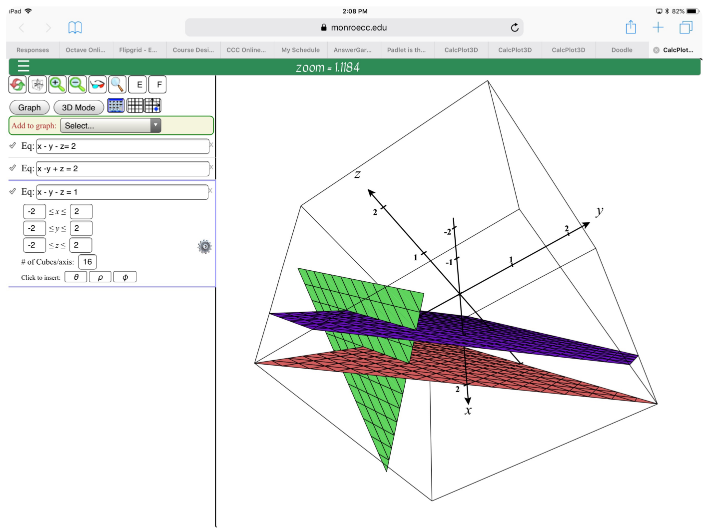

monroecc.edu
zoom = 1.1184
Graph
3D Mode
Add to graph:
Select...
Eq: x - y - z=0
Eq: x -y - z = -2
Eq: x- y- z= 2
# of Cubes/axis: 16
Click to insert:


monroecc.edu
zoom = 1.1184
Graph
3D Mode
Add to graph:
Select...
Eq: x - y - z= 2
Eq: x -y + z = 2
Eq: x - y - z = 1
# of Cubes/axis: 16
Click to insert:

OPERATIONS THAT PRODUCE EQUIVALENT SYSTEMS

Each of the following operations on a system of linear equations produces an equivalent system.

Add two equations.

Multiply an equation by a nonzero constant.

Add a multiple of an equation to another equation.

The evil plan is to get the system into row-echelon form.

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{33}x_3 &= b_3 \end{aligned}$$

DEFINITION OF A MATRIX

If *m* and *n* are positive integers, an $m \times n$ matrix (read m by n) ~~matrix~~ is a rectangular array

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

in which each entry, a_{ij}, of the matrix is a number. An $m \times n$ matrix has m rows and n columns.

Matrices are usually denoted by capital letters.

*The entry a_{ij} is located in the *i*th row and the *j*th column. The index *i* is called the row index because it identifies the row in which the entry lies, and the index *j* is called the column index because it identifies the column in which the entry lies.

**A matrix with *m* rows and *n* columns is said to be of size $m \times n$. When $m = n$, the matrix is called square of order *n* and the entries $a_{11}, a_{22}, a_{33}, \ldots$ are called the main diagonal entries.

THREE IMPORTANT TYPES OF MATRICES

1. Diagonal Matrices are square matrices with one's along the main diagonal, and zeros elsewhere. The main diagonal goes from the top left corner to the bottom right corner.
2. Coefficient Matrices are formed using the coefficients of the variables in systems of linear equations.
3. Augmented Matrices adjoin the coefficient matrix with the column matrix of constants.

Example 5: Consider the following system of linear equations.

$$\begin{aligned} x_1 - x_2 \quad + x_3 &= 2 \\ -x_1 + 3x_2 - 2x_3 &= 8 \\ 2x_1 + x_2 \quad - x_3 &= 1 \end{aligned}$$

a. Find the coefficient matrix (matrix of coefficients) and determine its size.

$$A = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 3 & -2 \\ 2 & 1 & -1 \end{bmatrix}, \text{ size } 3\times 3$$

b. Find the augmented matrix and determine its size.

$$B = [A \mid \vec{b}]$$

$$B = \left[\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ -1 & 3 & -2 & 8 \\ 2 & 1 & -1 & 1 \end{array}\right], \text{ size } 3\times 4$$

$$\vec{b} = \begin{bmatrix} 2 \\ 8 \\ 1 \end{bmatrix}$$

c. Solve the system and determine if it is consistent.

$$B = \left[\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ -1 & 3 & -2 & 8 \\ 2 & 1 & -1 & 1 \end{array}\right]$$

$R1 + R2 \to R2$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 0 & 2 & -1 & 10 \\ 2 & 1 & -1 & 1 \end{array}\right]$$

$-2R1 + R3 \to R3$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 0 & 2 & -1 & 10 \\ 0 & 3 & -3 & -3 \end{array}\right]$$

$\frac{1}{3}R3 \to 3$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 0 & 2 & -1 & 10 \\ 0 & 1 & -1 & -1 \end{array}\right]$$

$-\frac{1}{2}R2 + R3 \to 3$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 0 & 2 & -1 & 10 \\ 0 & 0 & -\frac{1}{2} & -6 \end{array}\right]$$

$$\begin{aligned} 1x_1 - 1x_2 + 1x_3 &= 2 \\ 2x_2 - 1x_3 &= 10 \\ -\tfrac{1}{2}x_3 &= -6 \end{aligned}$$

$x_3 = 12,\ x_2 = 11,\ x_1 = 1$

$\boxed{\{(1, 11, 12)\}}$

d. Check your result using [Octave](), which has the same commands as Matlab but is free☺.

i. Go to the very bottom of the page and enter the augmented matrix. I named the augmented matrix B. You use brackets to designate a matrix, use a __space__ between entries, and a __semicolon__ between rows.

```
» B = [[1 -1 1 2;-1 3 -2 8; 2 1 -1 1]
```

ii. After hitting "enter" the screen looks like this (you'll have a different command line number):

```
octave:18> B = [1 -1 1 2;-1 3 -2 8; 2 1 -1 1]
B =

   1  -1   1   2
  -1   3  -2   8
   2   1  -1   1
```

Now type in rref(B) to get the reduced row-echelon form of the augmented matrix:

```
octave:18> B = [1 -1 1 2;-1 3 -2 8; 2 1 -1 1]
B =

   1  -1   1   2
  -1   3  -2   8
   2   1  -1   1

» rref(B)
```

After hitting enter, you'll see:

```
octave:18> B = [1 -1 1 2;-1 3 -2 8; 2 1 -1 1]
B =

   1  -1   1   2
  -1   3  -2   8
   2   1  -1   1

octave:19> rref(B)
ans =

   1.00000   0.00000   0.00000    1.00000
   0.00000   1.00000   0.00000   11.00000
   0.00000   0.00000   1.00000   12.00000
```

iii. How should we interpret the results?

$1x_1 = 1$

$1x_2 = 11$

$1x_3 = 12$ $\rightarrow$

$\{(1, 11, 12)\}$

consistent system with independent equations

1.2 Gauss-Jordan Elimination

Learning Objectives

1. Use matrices and Gaussian elimination with back-substitution to solve a system of linear equations
2. Use matrices and Gauss-Jordan elimination to solve a system of linear equations
3. Solve a homogeneous system of linear equations
4. Fit a polynomial function to a set of data points
5. Set up and solve a system of equations to represent a network

Let's Do Our Math Stretches!

1. Interpret the following **augmented** matrices.

   a.

$$\begin{bmatrix} 1 & 0 & 0 & 8 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 5 \end{bmatrix} \rightarrow x_1 = 8,\ x_2 = 7,\ x_3 = 5$$

   b.

$$\begin{bmatrix} 1 & -1 & 0 & -2 \\ 0 & 1 & 3 & 11 \end{bmatrix} \rightarrow \begin{array}{l} x_1 - x_2 = -2 \rightarrow x_1 = x_2 - 2 = 9 - 3x_3 \\ x_2 + 3x_3 = 11 \rightarrow x_2 = 11 - 3x_3 \end{array}$$

Let $x_3 = t$, $x_2 = 11 - 3t$, $x_1 = 9 - 3t$

$$\boxed{\begin{array}{l} x_1 = 9 - 3t \\ x_2 = 11 - 3t,\ t \in \mathbb{R} \\ x_3 = t \end{array}}$$

   c.

$$\begin{bmatrix} 1 & 2 & 0 & 4 & 3 \\ 0 & 1 & 0 & 7 & 0 \end{bmatrix}$$

$x_1 = 3 + 10t$

$x_2 = -7t$

$\cancel{x_3 = 0}$ $\quad x_3 = s, \quad s, t \in \mathbb{R}$

$x_4 = t$

ELEMENTARY ROW OPERATIONS

1. Add two rows.
2. multiply a row by a nonzero constant.
3. Add a multiple of a row to another row.
4. Interchange (swap) any 2 rows.

Note: These operations also work for columns.

DEFINITION OF ROW-ECHELON FORM AND REDUCED ROW-ECHELON FORM

A matrix in row-echelon (ref) form has the following properties.

Any rows consisting entirely of zero occur at the bottom of the matrix. For each row that does not consist entirely of zeros, the first nonzero entry is 1 (called a leading 1). For two successive nonzero rows, the leading 1 in the higher row is farther to the left than the leading 1 in the lower row. A matrix in row-echelon form is in reduced row-echelon (rref) form when every column that has a leading 1 has zeros in every position above and below its leading 1.

Example 1: Determine which of the following augmented matrices are in row-echelon (ref) form.

a.
$$\left[\begin{array}{c|c} 1 & -\frac{1}{2} \end{array}\right]$$

yes !! :)

b.
$$\left[\begin{array}{ccc|c} 1 & -1 & 0 & 3 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 12 \end{array}\right]$$

yes - ref but not rref

c.
$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & -8 \\ 0 & 0 & 1 & 25 \\ 0 & 1 & 15 & -3 \end{array}\right]$$

no

GAUSS-JORDAN ELIMINATION

1. Write the augmented matrix of the system of linear equations.
2. Use elementary row operations to find an equivalent matrix in reduced row-echelon form. If this is not possible, write the equivalent system of equations and back substitute.
3. Interpret your results.

Example 2: Solve the system using Gauss-Jordan Elimination.

a.

$$x_1 + x_2 - 5x_3 = 3$$
$$x_1 - 2x_3 = 1$$
$$2x_1 - x_2 - x_3 = 0$$

$$B = \left[\begin{array}{ccc|c} 1 & 1 & -5 & 3 \\ 1 & 0 & -2 & 1 \\ 2 & -1 & -1 & 0 \end{array}\right]$$

$-R1 + R2 \to R2$

$$\left[\begin{array}{ccc|c} 1 & 1 & -5 & 3 \\ 0 & -1 & 3 & -2 \\ 2 & -1 & -1 & 0 \end{array}\right]$$

$-2R1 + R3 \to R3$

$$\left[\begin{array}{ccc|c} 1 & 1 & -5 & 3 \\ 0 & -1 & 3 & -2 \\ 0 & -3 & 9 & -6 \end{array}\right]$$

$-3R2 + R3 \to R3$

$$\left[\begin{array}{ccc|c} 1 & 1 & -5 & 3 \\ 0 & -1 & 3 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$x_1 + x_2 - 5x_3 = 3$$
$$-x_2 + 3x_3 = -2$$
$$0 = 0 \text{ true!! ☺}$$

$$x_1 = 3 - x_2 + 5x_3$$
$$x_2 = 3x_3 + 2$$
$$x_1 = 3 - (3x_3 + 2) + 5x_3$$
$$x_1 = 2x_3 + 1 = 2t + 1$$
$$x_2 = 3t + 2$$
$$x_3 = t$$

$$\boxed{\{(2t+1, 3t+2, t) : t \in \mathbb{R}\} \text{ consistent system with dependent equations}}$$

b.

$$5x_1 - 3x_2 + 2x_3 = 3$$
$$2x_1 + 4x_2 - x_3 = 7$$
$$x_1 - 11x_2 + 4x_3 = 3$$

$$B = \left[\begin{array}{ccc|c} 5 & -3 & 2 & 3 \\ 2 & 4 & -1 & 7 \\ 1 & -11 & 4 & 3 \end{array}\right]$$

$-2R_1 + 5R_2 \to R_2$

$$\left[\begin{array}{ccc|c} 5 & -3 & 2 & 3 \\ 0 & 26 & -9 & 29 \\ 1 & -11 & 4 & 3 \end{array}\right]$$

$-R_1 + 5R_3 \to R_3$

$$\left[\begin{array}{ccc|c} 5 & -3 & 2 & 3 \\ 0 & 26 & -9 & 29 \\ 0 & -52 & 18 & 12 \end{array}\right]$$

$2R_2 + R_3 \to R_3$

$$\left[\begin{array}{ccc|c} 5 & -3 & 2 & 3 \\ 0 & 26 & -9 & 29 \\ 0 & 0 & 0 & 70 \end{array}\right]$$

$$5x_1 - 3x_2 + 2x_3 = 3$$
$$26x_2 - 9x_3 = 29$$
$$0 = 70$$

FALSE! No solution

$\{ \}$, an inconsistent system with independent equations.

DEFINITION OF HOMOGENEOUS SYSTEMS OF LINEAR EQUATIONS

Systems of equations in which each of the constant terms is zero are called homogeneous. A homogeneous system of *m* equations in *n* variables has the form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots a_{2n}x_n &= 0 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots a_{3n}x_n &= 0 \\ \vdots \qquad \vdots \qquad \vdots \qquad \vdots \quad &\ \ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots a_{mn}x_n &= 0 \end{aligned}$$

****Homogenous linear systems either have the trivial solution, or infinitely many solutions**

* Trivial solution: $x_1 = x_2 = \cdots = x_n = 0$

$$\begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array}$$

Example 3: Solve the homogeneous linear system corresponding to the given coefficient matrix.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{aligned} x_1 \qquad &= 0 \\ x_2 + x_3 &= 0 \end{aligned}$$

$x_1 = 0, \; x_2 = -x_3 = -s, \; x_3 = s, \; x_4 = t$

$\{(0, -s, s, t) : s, t \in \mathbb{R}\}$ consistent system w/dependent equations

THEOREM 1.1: THE NUMBER OF SOLUTIONS OF A HOMOGENEOUS SYSTEM

Every homogeneous system of linear equations is __consistent__. If the system has fewer equations than variables, then it must have __infinitely many__ solutions.

POLYNOMIAL CURVE FITTING

Suppose *n* points in the *xy*-plane represent a collection of __data__ and you are asked to find a __polynomial__ function of degree __n-1__ whose graph passes through the specified points. This is called __polynomial__ __curve__ __fitting__. If all *x*-coordinates are distinct, then there is precisely __one__ polynomial function of degree $n-1$ (or less) that fits the *n* points. To solve for the *n* __coefficients__ of $p(x)$, __substitute__ each of the *n* points into the polynomial function and obtain *n* __linear__ equations in __n__ variables $a_0, a_1, a_2, \ldots, a_{n-1}$.

$$\begin{aligned} a_0 + a_1x_1 + a_2x_1^{\,2} + \cdots a_{n-1}x_1^{\,n-1} &= y_1 \\ a_0 + a_1x_2 + a_2x_2^{\,2} + \cdots a_{n-1}x_2^{\,n-1} &= y_2 \\ a_0 + a_1x_3 + a_2x_3^{\,2} + \cdots a_{n-1}x_3^{\,n-1} &= y_3 \\ \vdots \quad\;\; \vdots \quad\;\; \vdots \qquad\quad \vdots \quad &\quad\; \vdots \\ a_0 + a_1x_n + a_2x_n^{\,2} + \cdots a_{n-1}x_n^{\,n-1} &= y_n \end{aligned}$$

Example 4: Determine the polynomial function whose graph passes through the points, and graph the polynomial function, showing the given points.

$(1,8),(3,26),(5,60)$

$n = 3$ because we have 3 ordered pairs

$n - 1 = 2$

$p(x) = a_0 + a_1 x + a_2 x^2$

$p(1) = 8 = a_0 + a_1(1) + a_2(1)^2 = a_0 + a_1 + a_2$

$p(3) = 26 = a_0 + a_1(3) + a_2(3)^2 = a_0 + 3a_1 + 9a_2$

$p(5) = 60 = a_0 + a_1(5) + a_2(5)^2 = a_0 + 5a_1 + 25a_2$

$a_0 + a_1 + a_2 = 8$

$a_0 + 3a_1 + 9a_2 = 26$

$a_0 + 5a_1 + 25a_2 = 60$

$$B = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 8 \\ 1 & 3 & 9 & 26 \\ 1 & 5 & 25 & 60 \end{array}\right]$$

$-R1 + R2 \to R2$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 8 \\ 0 & 2 & 8 & 18 \\ 1 & 5 & 25 & 60 \end{array}\right]$$

$-R1 + R3 \to R3$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 8 \\ 0 & 1 & 4 & 9 \\ 0 & 4 & 24 & 52 \end{array}\right]$$

$-4R2 + R3 \to R$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 8 \\ 0 & 1 & 4 & 9 \\ 0 & 0 & 8 & 16 \end{array}\right]$$

$a_0 + a_1 + a_2 = 8$

$a_1 + 4a_2 = 9$

$8a_2 = 16$

$a_2 = 2, \; a_1 = 1, \; a_0 = 5$

$p(x) = 5 + x + 2x^2$

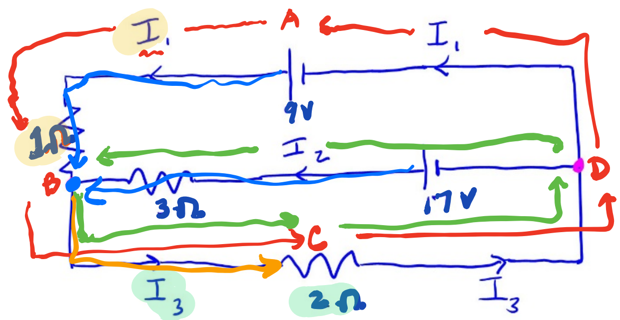
NETWORK ANALYSIS

Networks composed of branches and junctions are used as models in fields like economics, traffic analysis, and electrical engineering. In an electrical network model. you use Kirchoff's Laws to find the system of equations.

Kirchoff's Laws

1. Junctions: All the current flowing into a junction must flow out of it.
2. Paths: The sum of the *IR* terms, where *I* denotes current and *R* denotes resistence in any direction around a closed path is equal to the total voltage in the path in that direction.

Example 5: Determine the currents in the various branches of the electrical network. The units of current are amps and the units of resistance are ohms.



PATH: ABCDA

$1I_1 + 2I_3 = 9$

current

$I_1 + I_2 = I_3$

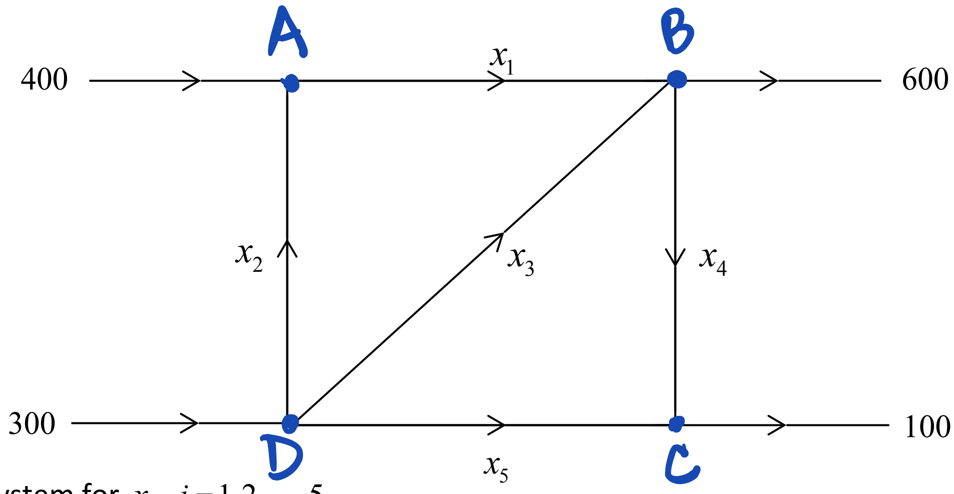
PATH: BCDB

$2I_3 + 3I_2 = 17$

$$\begin{aligned} I_1 \qquad + 2I_3 &= 9 \\ I_1 + I_2 - I_3 &= 0 \\ 3I_2 + 2I_3 &= 17 \end{aligned} \rightarrow \boxed{\begin{aligned} I_1 &= 1A \\ I_2 &= 3A \\ I_3 &= 4A \end{aligned}}$$

A denotes amperes

Example 6: The figure below shows the flow of traffic through a network of streets.



Solve this system for x_i, $i = 1, 2, \ldots, 5$.

$$\left.\begin{aligned} 400 + x_2 &= x_1 \\ x_1 + x_3 &= 600 + x_4 \\ x_4 + x_5 &= 100 \\ 300 &= x_2 + x_3 + x_5 \end{aligned}\right\} \quad \begin{aligned} x_1 - x_2 \qquad\qquad &= 400 \\ x_1 \qquad + x_3 - x_4 \quad &= 600 \\ x_4 + x_5 &= 100 \\ x_2 + x_3 \qquad + x_5 &= 300 \end{aligned}$$

$$\left[\begin{array}{ccccc|c} 1 & -1 & 0 & 0 & 0 & 400 \\ 1 & 0 & 1 & -1 & 0 & 600 \\ 0 & 0 & 0 & 1 & 1 & 100 \\ 0 & 1 & 1 & 0 & 1 & 300 \end{array}\right] \to \left[\begin{array}{ccccc|c} 1 & 0 & 1 & 0 & 1 & 700 \\ 0 & 1 & 1 & 0 & 1 & 300 \\ 0 & 0 & 0 & 1 & 1 & 100 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Find the traffic flow when $x_3 = 0$ and $x_5 = 100$.

$x_1 = 700 - 0 - 100 = 600$

$x_2 = 300 - 0 - 100 = 200$

$x_3 = 0$

$x_4 = 100 - 100 = 0$ and $x_5 = 100$

Find the traffic flow when $x_3 = x_5 = 100$.

$x_1 = 700 - 100 - 100 = 500$

$x_2 = 300 - 100 - 100 = 100$

$x_4 = 100 - 100 = 0$ and $x_3 = x_5 = 100$

$x_1 = 700 - x_3 - x_5$

$x_2 = 300 - x_3 - x_5$

$x_4 = 100 - x_5$

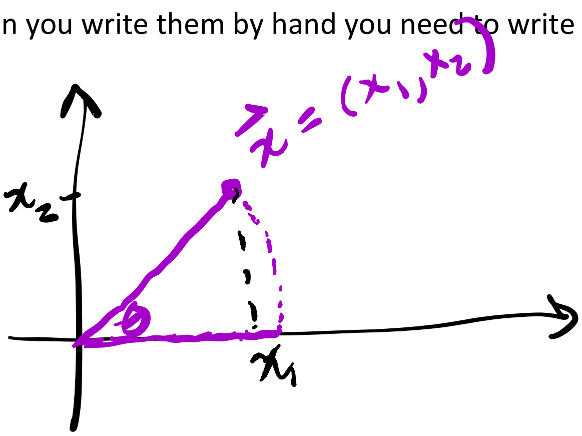
1.3 The Vector Space R^n

Learning Objectives

1. Perform basic vector operations in R^2 and represent them graphically
2. Perform basic vector operations in R^n
3. Write a vector as a linear combination of other vectors
4. Perform basic operations with column vectors
5. Determine whether one vector can be written as a linear combination of 2 or more vectors
6. Determine if a subset of R^n is a subspace of R^n

VECTORS IN THE PLANE

A vector is characterized by two quantities, length and direction, and is represented by a directed line segment. Geometrically, a vector in the plane is represented by a directed line segment with its initial point at the origin and its terminal point at (x_1, x_2). Boldface lowercase letters often designate vectors when you're using a computer, but when you write them by hand you need to write an arrow above the letter designating the vector.

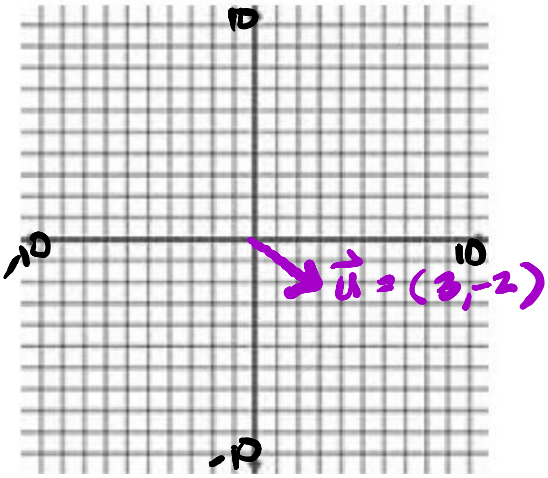


The same ordered pair used to represent its terminal point also represents the vector. That is, $\vec{x} = (x_1, x_2)$. The coordinates x_1 and x_2 are called the components of the vector $\mathbf{x}$. Two vectors in the plane $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$ are equal if and only if $u_1 = v_1$ and $u_2 = v_2$. What do you think the zero vector is for R^2? $\vec{0} = (0,0)$ How about R^3? $\vec{0} = (0,0,0)$ R^6? $\vec{0} = (0,0,0,0,0,0)$

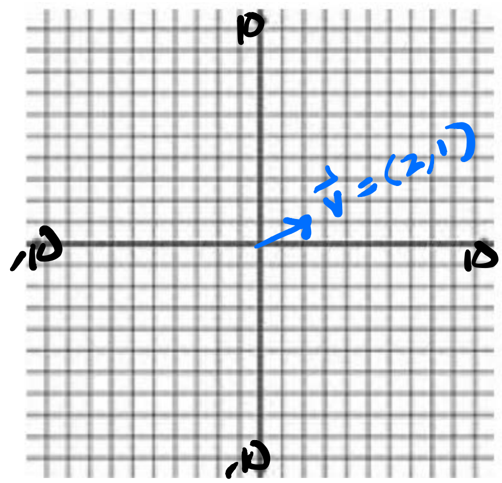
R^n? $\vec{0} = \underbrace{(0,0,0,\ldots,0)}_{n \text{ zero components}}$

Example 1: Use a directed line segment to represent the vector, and give the graphical representation of the vector operations.

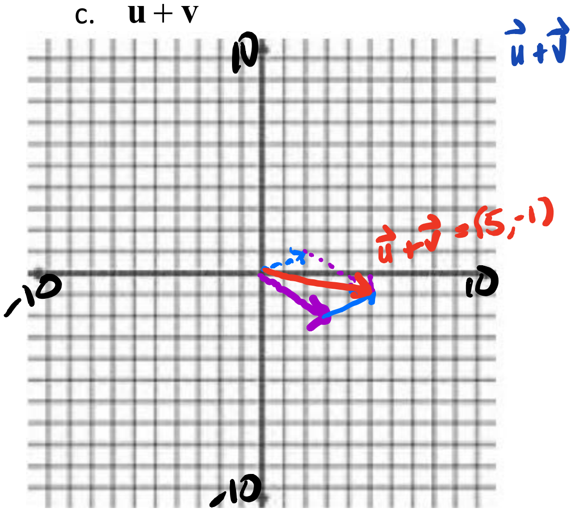
a. $\mathbf{u} = (3, -2)$



b. $\mathbf{v} = (2, 1)$

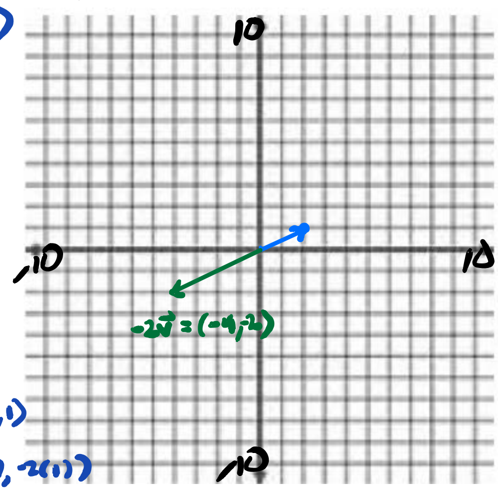


c. $\mathbf{u} + \mathbf{v}$



$\vec{u} + \vec{v} = (3, -2) + (2, 1)$
$= (3+2, -2+1)$
$= (5, -1)$

d. $-2\mathbf{v}$



$-2\vec{v} = -2(2, 1)$
$= (-2(2), -2(1))$
$= (-4, -2)$

IMPORTANT VECTOR SPACES

R = 1-space = the set of real numbers.
R^2 = 2-space = the set of all ordered pairs of real numbers.
R^3 = 3-space = the set of all ordered triples of real numbers.
R^n = n-space = the set of all ordered n-tuples of real numbers.

DEFINITION OF VECTOR ADDITION AND SCALAR MULTIPLICATION [R^n]

Let $\vec{u} = (u_1, u_2, \ldots, u_n)$ and $\vec{v} = (v_1, v_2, \ldots, v_n)$ be vectors in R^n, and let $c \in R$. Then the sum of $\vec{u}$ and $\vec{v}$ is defined as the vector, $\vec{u} + \vec{v} = (u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n)$ and the scalar multiplication of $\vec{u}$ by c is defined as the vector, $c\vec{u} = (cu_1, cu_2, \ldots, cu_n)$.

Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors in R^n, and let c and d be scalars.

Let $\vec{u} = (u_1, u_2, \ldots, u_n)$, $\vec{v} = (v_1, v_2, \ldots, v_n)$, $\vec{w} = (w_1, w_2, \ldots, w_n)$, v_i, u_i, w_i, $i = 1, 2, \ldots, n$ $\in R$, and $c, d \in R$.

ADDITION:

1. $\mathbf{u}+\mathbf{v}$ is a vector in R^n. — closure

Proof:

2. $\mathbf{u}+\mathbf{v} = \vec{v}+\vec{u}$ — commutative property

Proof:

$$\vec{u}+\vec{v} = (u_1, u_2, \ldots, u_n) + (v_1, v_2, \ldots, v_n)$$
$$= (u_1+v_1, u_2+v_2, \ldots, u_n+v_n) \quad \text{defn. vector +}$$
$$= (v_1+u_1, v_2+u_2, \ldots, v_n+u_n) \quad R \text{ is comm } (+)$$
$$= (v_1, v_2, \ldots, v_n) + (u_1, u_2, \ldots, u_n) \quad \text{defn vect. +}$$
$$= \vec{v}+\vec{u} \;//$$

3. $\mathbf{u}+(\mathbf{v}+\mathbf{w}) = (\vec{u}+\vec{v})+\vec{w}$ — Associative property

4. $\mathbf{u}+\mathbf{0} = \vec{u}$ — additive identity property

5. $\mathbf{u}+(-\mathbf{u}) = \vec{0}$ — additive inverse property

SCALAR MULTIPLICATION:

6. $c\mathbf{u}$ is a vector in ~~the~~ R^n. — closure

7. $c(\mathbf{u}+\mathbf{v}) = c\vec{u}+c\vec{v}$ — distributive property

Proof:

$$c(\vec{u}+\vec{v}) = c[(u_1, u_2, \ldots, u_n) + (v_1, v_2, \ldots, v_n)]$$
$$= c(u_1+v_1, u_2+v_2, \ldots, u_n+v_n) \quad \text{defn vec. +}$$
$$= (c(u_1+v_1), c(u_2+v_2), \ldots, c(u_n+v_n)) \quad \text{defn vec. scal. mult.}$$
$$= (cu_1+cv_1, cu_2+cv_2, \ldots, cu_n+cv_n) \quad R \text{ distributes}$$

see page below...

8. $(c+d)\mathbf{u} = c\vec{u}+d\vec{u}$ — distributive property

9. $c(d\mathbf{u}) = (cd)\vec{u}$ — associative property

10. $1(\mathbf{u}) = \vec{u}$ — multiplicative identity property

$$= (cu_1, cu_2, \ldots, cu_n) + (cv_1, cv_2, \ldots, cv_n)$$ defn of vec. +

$$= c(u_1, u_2, \ldots, u_n) + c(v_1, v_2, \ldots, v_n)$$ defn of vec. scal. mult

$$= c\vec{u} + c\vec{v} \quad //$$

Example 2: Solve for $\mathbf{w}$, where $\mathbf{u}=(2,-1,3,4)$, and $\mathbf{v}=(-1,8,0,3)$.

a. $\mathbf{w}+\mathbf{u}=-\mathbf{v}$

$\vec{w}+\vec{u}-\vec{u}=-\vec{v}-\vec{u}$

$\vec{w}+\vec{u}+(-\vec{u})=-1(\vec{v}+\vec{u})$

$\vec{w}+\vec{0}=-[(-1,8,0,3)+(2,-1,3,4)]$

$\vec{w}=-(1,7,3,7)$

$\boxed{\vec{w}=(-1,-7,-3,-7)}$

b. $\mathbf{w}+3\mathbf{v}=-2\mathbf{u}$

$\vec{w}=-2\vec{u}-3\vec{v}$

$\vec{w}=(-4,2,-6,-8)-(-3,24,0,9)$

$\boxed{\vec{w}=(-1,-22,-6,-17)}$

DEFINITION OF COLUMN VECTOR ADDITION AND SCALAR MULTIPLICATION

Let $u_1,u_2,\ldots,u_n$, $v_1,v_2,\ldots,v_n$, and c be scalars.

$$\begin{bmatrix}u_1\\u_2\\\vdots\\u_n\end{bmatrix}+\begin{bmatrix}v_1\\v_2\\\vdots\\v_n\end{bmatrix}=\begin{bmatrix}u_1+v_1\\u_2+v_2\\\vdots\\u_n+v_n\end{bmatrix} \text{ and } c\begin{bmatrix}u_1\\u_2\\\vdots\\u_n\end{bmatrix}=\begin{bmatrix}cu_1\\cu_2\\\vdots\\cu_n\end{bmatrix}$$

Example 3: Find the following, given that $\mathbf{u}=\begin{bmatrix}-3\\18\\-1\\31\\-9\end{bmatrix}$, and $\mathbf{v}=\begin{bmatrix}-2\\41\\-6\\-3\\15\end{bmatrix}$.

a. $2\mathbf{u}-3\mathbf{v}$

$=\begin{bmatrix}-6\\36\\-2\\62\\-18\end{bmatrix}+\begin{bmatrix}6\\-123\\18\\9\\-45\end{bmatrix}\rightarrow=\boxed{\begin{bmatrix}0\\-87\\16\\71\\-63\end{bmatrix}}$

b. $-(\mathbf{v}+\mathbf{u})$

$=-\begin{bmatrix}-5\\59\\-7\\28\\6\end{bmatrix}$

$=\boxed{\begin{bmatrix}5\\-59\\7\\-28\\-6\end{bmatrix}}$

THEOREM 1.3: PROPERTIES OF ADDITIVE IDENTITY AND ADDITIVE INVERSE

Let $\mathbf{v}$ be a vector in R^n, and let c be a scalar. Then the following properties are true.

1. The additive identity is unique.

Proof:

Suppose $\exists \vec{u} \in R^n \ni \vec{v}+\vec{u}=\vec{v}$.

$(\vec{v}+\vec{u})+(-\vec{v}) = \vec{v}+(-\vec{v})$

$\vec{u}+[\vec{v}+(-\vec{v})] = \vec{0}$

$\vec{u}+\vec{0} = \vec{0}$

$\vec{u} = \vec{0}$

$\therefore$ The additive identity is unique. //

2. The additive inverse is unique.
3. $0\mathbf{v} = \vec{0}$
4. $c\mathbf{0} = \vec{0}$
5. If $c\mathbf{v}=\mathbf{0}$, then $c=0$ or $\vec{v}=\vec{0}$.
6. $-(-\mathbf{v}) = \vec{v}$

LINEAR COMBINATIONS OF VECTORS

An important type of problem in linear algebra involves writing one vector as the sum of scalar multiples of other vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$. The vector $\vec{x}$, $\vec{x}=c_1\vec{v}_1+c_2\vec{v}_2+\cdots+c_n\vec{v}_n$ is called a linear combination of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$.

Example 4: If possible, write $\mathbf{u}$ as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$, where $\mathbf{v}_1=(1,2)$ and $\mathbf{v}_2=(-1,3)$.

a. $\mathbf{u}=(0,3)$

Let's check

$\vec{u}=c_1\vec{v}_1+c_2\vec{v}_2 \rightarrow (0,3)=\frac{3}{5}(1,2)+\frac{3}{5}(-1,3)$

$(0,3)=c_1(1,2)+c_2(-1,3)$

$(0,3)=(c_1,2c_1)+(-c_2,3c_2)$

$(0,3)=(c_1-c_2,2c_1+3c_2)$

b. $\mathbf{u}=(1,-1)$

see next page

$c_1-c_2=0 \rightarrow c_1=c_2=\frac{3}{5}$

$2c_1+3c_2=3 \rightarrow 2c_2+3c_2=3 \rightarrow 5c_2=3 \rightarrow c_2=\frac{3}{5}$

b) $\vec{u} = \langle 1,-1 \rangle$, $\vec{v}_1 = (1,2)$, $\vec{v}_2 = (-1,3)$

$$c_1\vec{v}_1 + c_2\vec{v}_2 = \vec{u}$$

$$c_1(1,2) + c_2(-1,3) = (1,-1)$$

$$c_1 - c_2 = 1$$
$$2c_2 + 3c_2 = -1$$

$$B = \left[\begin{array}{cc|c} 1 & -1 & 1 \\ 2 & 3 & -1 \end{array}\right]$$

$\downarrow$

$-2R1 + R2 \to R2$

$$\left[\begin{array}{cc|c} 1 & -1 & 1 \\ 0 & 5 & -3 \end{array}\right]$$

$\downarrow$

$5R1 + R2 \to R1$

$$\left[\begin{array}{cc|c} 5 & 0 & 2 \\ 0 & 5 & -3 \end{array}\right]$$

$\downarrow$

$$\left[\begin{array}{cc|c} 1 & 0 & 2/5 \\ 0 & 1 & -3/5 \end{array}\right]$$

$$c_1 = \frac{2}{5}$$

$$c_2 = -\frac{3}{5}$$

$$\frac{2}{5}(1,2) - \frac{3}{5}(-1,3) = (1,-1)$$

$$S = \{\vec{v}_1, \vec{v}_2\}$$

If you can obtain any vector in R^2 using a linear combination of the vectors in S, the S is a spanning set of R^2.

Example 5: If possible, write $\mathbf{u}$ as a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$, where $\mathbf{v}_1 = (1,3,5)$, $\mathbf{v}_2 = (2,-1,3)$, and $\mathbf{v}_3 = (-3,2,-4)$.

$\mathbf{u} = (-1,7,2)$

$$c_1(1,3,5)+c_2(2,-1,3)+c_3(-3,2,-4)=(-1,7,2)$$

$$\begin{aligned} c_1+2c_2-3c_3 &= -1\\ 3c_1-c_2+2c_3 &= 7\\ 5c_1+3c_2-4c_3 &= 2 \end{aligned}$$

inconsistent system

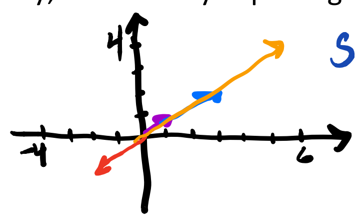
It is not possible to write $\vec{u}$ as a linear combination of the vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$.

WHAT THE HECK DOES IT ALL MEAN??

Any vector space consists of __4__ entities: a __set__ of __vectors__, a set of __scalars__, and __2__ operations. Currently, we are only exploring the vector spaces __in R^n__.

Let's think about the following subset of R^2:

$$S = \left\{ \left(x, \frac{1}{2}x \right) : x \in \mathbb{R} \right\}$$



S spans the line $y=\frac{1}{2}x$

Is the set S a vector space? Let's find out!

Let $\vec{u}=(u_1,\frac{1}{2}u_1)$, $\vec{v}=(v_1,\frac{1}{2}v_1)$, $\vec{w}=(w_1,\frac{1}{2}w_1)$, and $u_1, v_1, w_1, c, d \in R$.

1. Closure under addition.

$\vec{u}$ and $\vec{v} \in S$.

$$\begin{aligned}\vec{u}+\vec{v} &= (u_1,\tfrac{1}{2}u_1)+(v_1,\tfrac{1}{2}v_1)\\ &= (u_1+v_1, \tfrac{1}{2}u_1+\tfrac{1}{2}v_1) \quad \text{defn. vect. +}\\ &= (u_1+v_1, \tfrac{1}{2}(u_1+v_1)) \quad R \text{ is dist.}\end{aligned}$$

//

2. Commutativity under addition.

$$\begin{aligned}\vec{u}+\vec{v} &= (u_1,\tfrac{1}{2}u_1)+(v_1,\tfrac{1}{2}v_1)\\ &= (u_1+v_1, \tfrac{1}{2}u_1+\tfrac{1}{2}v_1) \quad \text{defn vect +}\\ &= (v_1+u_1, \tfrac{1}{2}v_1+\tfrac{1}{2}u_1) \quad R \text{ is comm.}\\ &= (v_1,\tfrac{1}{2}v_1)+(u_1,\tfrac{1}{2}u_1) \quad \text{defn vect +}\\ &= \vec{v}+\vec{u}\end{aligned}$$

//

3. Associativity under addition.

$$\vec{u}+(\vec{v}+\vec{w}) = (u_1,\tfrac{1}{2}u_1)+\left[(v_1,\tfrac{1}{2}v_1)+(w_1,\tfrac{1}{2}w_1)\right]$$

$$= (u_1,\tfrac{1}{2}u_1)+(v_1+w_1,\tfrac{1}{2}v_1+\tfrac{1}{2}w_1) \text{ - defn vect +}$$

$$= (u_1+(v_1+w_1),\ \tfrac{1}{2}u_1+(\tfrac{1}{2}v_1+\tfrac{1}{2}w_1))$$

$$= ((u_1+v_1)+w_1,\ (\tfrac{1}{2}u_1+\tfrac{1}{2}v_1)+\tfrac{1}{2}w_1) \quad \mathbb{R} \text{ is assoc } (+)$$

$$= (u_1+v_1,\ \tfrac{1}{2}u_1+\tfrac{1}{2}v_1)+(w_1,\tfrac{1}{2}w_1) \text{ defn vect +}$$

$$= \left[(u_1,\tfrac{1}{2}u_1)+(v_1,\tfrac{1}{2}v_1)\right]+\vec{w} \longrightarrow = (\vec{u}+\vec{v})+\vec{w} \quad //$$

4. Additive identity.

$$\vec{u}+\vec{0} = (u_1,\tfrac{1}{2}u_1)+(0,\tfrac{1}{2}\cdot 0)$$

$$= (u_1+0,\ \tfrac{1}{2}u_1+0) \text{ defn vect +}$$

$$= (u_1,\tfrac{1}{2}u_1) \text{ add. identity prop for } \mathbb{R}$$

$$= \vec{u} \quad //$$

5. Additive inverse.

$$\vec{u}+(-\vec{u}) = (u_1,\tfrac{1}{2}u_1)+\left[-(u_1,\tfrac{1}{2}u_1)\right]$$

$$= (u_1,\tfrac{1}{2}u_1)+(-u_1,-(\tfrac{1}{2}u_1)) \text{ defn vect. scal. mult.}$$

$$= (u_1+(-u_1),\ \tfrac{1}{2}u_1+(-\tfrac{1}{2}u_1)) \text{ defn vect. +}$$

add inv. prop. for R

$$= (0,\ \tfrac{1}{2}(u_1+(-u_1)))$$

$$= (0,\tfrac{1}{2}\cdot 0) \rightarrow = \vec{0}$$

6. Closure under scalar multiplication.

$$c\vec{u} = c(u_1,\tfrac{1}{2}u_1)$$

$$= (cu_1,\ c(\tfrac{1}{2}u_1)) \text{ defn vect. scal. mult.}$$

$$= (cu_1,\ \tfrac{1}{2}(cu_1)) \quad \mathbb{R} \text{ is comm } (\times)$$

which $\in S$ //

7. Distributivity under scalar multiplication (2 vectors and 1 scalar).

$$c(\vec{u}+\vec{v}) = c\left[(u_1, \tfrac{1}{2}u_1) + (v_1, \tfrac{1}{2}v_1)\right]$$

$$= c\left(u_1+v_1, \tfrac{1}{2}u_1+\tfrac{1}{2}v_1\right) \quad \text{defn vect. +}$$

$$= \left(c(u_1+v_1), c(\tfrac{1}{2}u_1+\tfrac{1}{2}v_1)\right) \quad \text{defn vect scal. mult.}$$

$$= \left(cu_1+cv_1, c(\tfrac{1}{2}u_1)+c(\tfrac{1}{2}v_1)\right) \quad \mathbb{R} \text{ is dist.}$$

$$= \left(cu_1, c(\tfrac{1}{2}u_1)\right) + \left(cv_1, c(\tfrac{1}{2}v_1)\right) \quad \text{defn vect +}$$

$$= c(u_1, \tfrac{1}{2}u_1) + c(v_1, \tfrac{1}{2}v_1) \quad \text{defn vect. scal. mult}$$

$$= c\vec{u} + c\vec{v} \quad //$$

8. Distributivity under scalar multiplication (2 scalars and 1 vector).

$$(c+d)\vec{u} = (c+d)(u_1, \tfrac{1}{2}u_1)$$

$$= \left((c+d)u_1, (c+d)(\tfrac{1}{2}u_1)\right) \quad \text{defn vect scal. mult}$$

$$= \left(cu_1+du_1, c(\tfrac{1}{2}u_1)+d(\tfrac{1}{2}u_1)\right) \quad \mathbb{R} \text{ is dist.}$$

$$= \left(cu_1, c(\tfrac{1}{2}u_1)\right) + \left(du_1, d(\tfrac{1}{2}u_1)\right) \quad \text{defn. vect +}$$

$$= c(u_1, \tfrac{1}{2}u_1) + d(u_1, \tfrac{1}{2}u_1) \quad \text{defn. vect. scal. mult.}$$

$$= c\vec{u} + d\vec{u} \quad //$$

9. Associativity under scalar multiplication.

$$c(d\vec{u}) = c\left[d\left(u_1, \tfrac{1}{2}u_1\right)\right]$$

$$= c\left(du_1, d\left(\tfrac{1}{2}u_1\right)\right) \leftarrow \text{defn vect. scal. mult.}$$

$$= \left(c(du_1), c\left[d\left(\tfrac{1}{2}u_1\right)\right]\right)$$

$$= \left((cd)u_1, (cd)\left(\tfrac{1}{2}u_1\right)\right) \quad \mathbb{R} \text{ is assoc } (\times)$$

$$= (cd)\left(u_1, \tfrac{1}{2}u_1\right) \quad \text{defn vector scal. mult.}$$

$$= (cd)\vec{u} \; //$$

10. Scalar multiplicative identity.

$$1\vec{u} = 1\left(u_1, \tfrac{1}{2}u_1\right)$$

$$= \left(1(u_1), 1\left(\tfrac{1}{2}u_1\right)\right) \quad \text{defn vect. scalar mult}$$

$$= \left(u_1, \left(1 \cdot \tfrac{1}{2}\right)u_1\right) \leftarrow \mathbb{R} \text{ is associative}$$

$$= \left(u_1, \tfrac{1}{2}u_1\right) \leftarrow 1 \text{ is the mult. identity for } \mathbb{R}$$

$$= \vec{u} \; //$$

Conclusion?

$$S = \left\{\left(x, \tfrac{1}{2}x\right) : x \in \mathbb{R}\right\} \text{ is a vector space !!}$$

Example 6: Determine whether the set W is a vector space with the standard operations. Justify your answer.

$W = \{(x_1, x_2, 4) : x_1 \text{ and } x_2 \in \mathbb{R}\}$

$\vec{u} = (1,2,4)$ and $\vec{v} = (5,6,4) \in W$

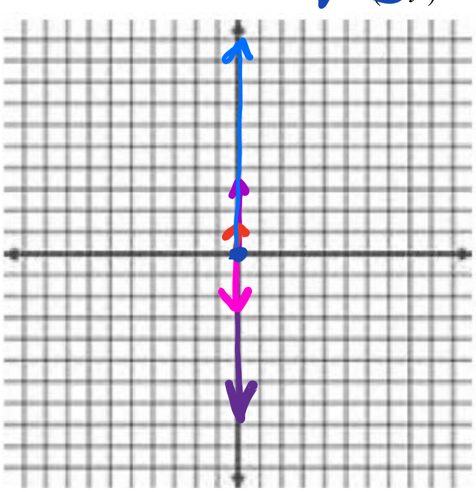
$\vec{u} + \vec{v} = (6,8,8) \notin W$ so W is not closed under addition. So... NOT a vector space.

SUBSPACES

In many applications of linear algebra, vector spaces occur as a __subspace__ of larger spaces. A __nonempty__ subset of a vector __space__ is a __subspace__ when it is a vector __space__ with the __same__ operations defined in the __original__ vector space.

Consider the following: ~~$W = (0, y)$~~ and $V = R^2$.

$W = \{(0,y) : y \in R\}$



$W \subseteq R^2$, subset

W is nonempty

$\vec{u} = (0, u_1)$, $\vec{v} = (0, v_1)$, $c, u_1, v_1 \in R$.

$\vec{u} + \vec{v} = (0, u_1) + (0, v_1)$
$= (0, u_1 + v_1) \in W$, so we have closure under addition.

$c\vec{u} = c(0, u_1)$
$= (c(0), c(u_1))$
$= (0, cu_1) \in W$, so we have closure under scal. mult.

$\therefore$ W is a subspace of R^2.

DEFINITION OF A SUBSPACE OF A VECTOR SPACE

A nonempty subset W of a vector space V is called a subspace of V when W is a vector space under the operations of addition and scalar multiplication defined in V.

THEOREM 1.4: TEST FOR A SUBSPACE

If W is a nonempty subset of a vector space V, then W is a subspace of V if and only if the following closure conditions hold.

1. If $\mathbf{u}$ and $\mathbf{v}$ are in W, then $\vec{u}+\vec{v}$ is in W.
2. If $\mathbf{u}$ is in W and c is any scalar, then $c\vec{u}$ is in W.

Example 7: Verify that W is a subspace of V.

$W = \{(x, y, 2x-3y) : x \text{ and } y \in \mathbb{R}\}$

$V = R^3$

$\{\ \}$, $\varnothing$ mean the empty or null set

1) $W \subseteq R^3$ ✓

2) W is non-empty

Let $\vec{u} = (u_1, u_2, 2u_1 - 3u_2)$, $\vec{v} = (v_1, v_2, 2v_1 - 3v_2)$, $u_1, u_2, v_1, v_2, c \in R$

3) $\vec{u} + \vec{v} = (u_1, u_2, 2u_1 - 3u_2) + (v_1, v_2, 2v_1 - 3v_2)$

$= (u_1+v_1, u_2+v_2, (2u_1-3u_2)+(2v_1-3v_2))$

$= (u_1+v_1, u_2+v_2, 2u_1+2v_1+(-3u_2-3v_2))$

$= (u_1+v_1, u_2+v_2, 2(u_1+v_1)-3(u_2+v_2)) \in W$ ✓

4) $c\vec{u} = c(u_1, u_2, 2u_1-3u_2)$

$= (cu_1, cu_2, c(2u_1-3u_2))$

$= (cu_1, cu_2, c(2u_1)-c(3u_2))$

$= (cu_1, cu_2, 2(cu_1)-3(cu_2)) \in W$ ✓

THEOREM 1.5: THE INTERSECTION OF TWO SUBSPACES IS A SUBSPACE

If V and W are both subspaces of a vector space U, then the intersection of V and W, denoted by $V \cap W$, is also a subspace of U.

1.4 Basis and Dimension of R^n

Learning Objectives

1. Determine if a set of vectors in R^n spans R^n.
2. Determine if a set of vectors in R^n is linearly independent
3. Determine if a set of vectors in R^n is a basis for R^n
4. Find standard bases for R^n
5. Determine the dimension of R^n

Let's do our math stretches!

If possible, write the vector $\mathbf{z} = (-4, -3, 3)$ as a linear combination of the vectors in $S = \{\underbrace{(1, 2, -2)}_{\vec{v}_1}, \underbrace{(2, -1, 1)}_{\vec{v}_2}\}$.

$$\vec{z} = c_1\vec{v}_1 + c_2\vec{v}_2$$

$$(-4, -3, 2) = c_1(1, 2, -2) + c_2(2, -1, 1)$$

$$-4 = c_1 + 2c_2$$
$$-3 = 2c_1 - c_2$$
$$3 = -2c_1 + c_2$$

$$\left[\begin{array}{cc|c} 1 & 2 & -4 \\ 2 & -1 & -3 \\ -2 & 1 & 3 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{array}\right]$$

What if...

$$S = \{(1, 2, -2), (2, -1, 1), (-4, -3, 3)\}$$

DEFINITION OF LINEAR COMBINATION OF VECTORS IN A VECTOR SPACE

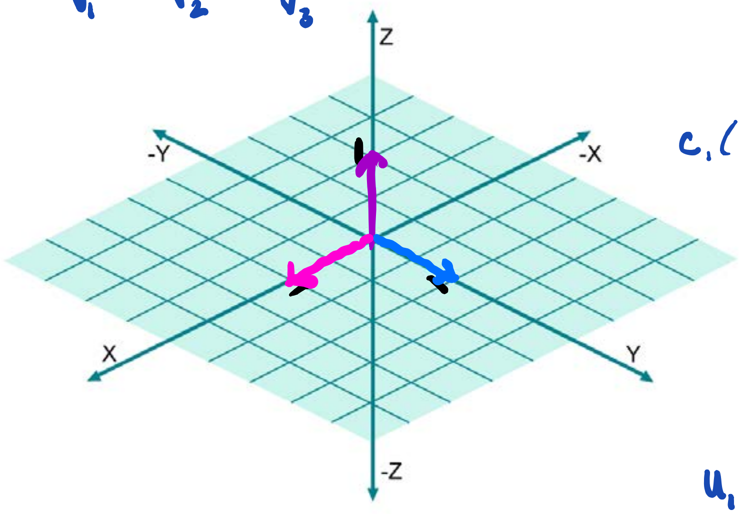
A vector $\mathbf{v}$ in a vector space V is called a linear combination of the vectors $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k$ in V when $\mathbf{v}$ can be written in the form $\vec{v} = c_1\vec{u}_1 + c_2\vec{u}_2 + \cdots + c_k\vec{u}_k$

where $c_1, c_2, \ldots c_k$ are scalars, $\in R$

DEFINITION OF A SPANNING SET OF A VECTOR SPACE

Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ be a subset of a vector space V. The set S is called a spanning set of V when every vector in V can be written as a linear combination of vectors in S.

$S = \{\underbrace{(1,0,0)}_{\vec{v}_1}, \underbrace{(0,1,0)}_{\vec{v}_2}, \underbrace{(0,0,1)}_{\vec{v}_3}\}$, $V = R^3$



Let $\vec{u} = (u_1, u_2, u_3)$ be any vector in R^3. So $u_1, u_2, u_3 \in R$.

$c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{u}$

$c_1(1,0,0) + c_2(0,1,0) + c_3(0,0,1) = (u_1, u_2, u_3)$

$c_1 = u_1$

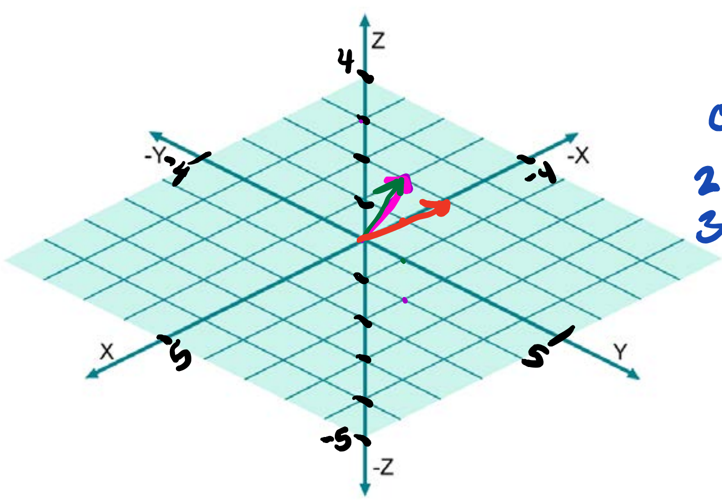
$c_2 = u_2$

$c_3 = u_3$

$u_1(1,0,0) + u_2(0,1,0) + u_3(0,0,1) = (u_1, u_2, u_3)$

So S spans R^3.

$S = \{\underbrace{(1,2,3)}_{\vec{v}_1}, \underbrace{(0,1,2)}_{\vec{v}_2}, \underbrace{(-1,1,1)}_{\vec{v}_3}\}$, $V = R^3$



Let $\vec{u} = (u_1, u_2, u_3) \ni u_i, i = 1,2,3 \in R$

$c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{u}$

$c_1(1,2,3) + c_2(0,1,2) + c_3(-1,1,1) = (u_1, u_2, u_3)$

$$\begin{aligned} c_1 \qquad\quad - c_3 &= u_1 \\ 2c_1 + c_2 + c_3 &= u_2 \\ 3c_1 + 2c_2 + c_3 &= u_3 \end{aligned}$$

$\boxed{S \text{ is a spanning set of } R^3}$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & u_1 \\ 2 & 1 & 1 & u_2 \\ 3 & 2 & 1 & u_3 \end{array}\right]$$

$-2R1 + R2 \to R2$

$\downarrow$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & u_1 \\ 0 & 1 & 3 & -2u_1 + u_2 \\ 3 & 2 & 1 & u_3 \end{array}\right]$$

$-3R1 + R3 \to R3$

$\downarrow$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & u_1 \\ 0 & 1 & 3 & -2u_1 + u_2 \\ 0 & 2 & 4 & -3u_1 + u_3 \end{array}\right]$$

Tired brain...

$$\begin{array}{l} 4u_1 - 2u_2 \\ -3u_1 + \quad u_3 \\ \hline u_1 - 2u_2 + u_3 \end{array}$$

$-2R2 + R3 \to R3$

$\downarrow$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & u_1 \\ 0 & 1 & 3 & -2u_1 + u_2 \\ 0 & 0 & -2 & u_1 - 2u_2 + u_3 \end{array}\right]$$

$3u_1 - 6u_2 + 3u_3$

$-4u_1 + 2u_2$

$3R3 + 2R2 \to R2$

$\downarrow$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & u_1 \\ 0 & 2 & 0 & -u_1 - 4u_2 + 3u_3 \\ 0 & 0 & -2 & u_1 - 2u_2 + u_3 \end{array}\right]$$

$-2R1 + R3 \to R1$

$\downarrow$

$$\left[\begin{array}{ccc|c} -2 & 0 & 0 & -u_1 - 2u_2 + u_3 \\ 0 & 2 & 0 & -u_1 - 4u_2 + 3u_3 \\ 0 & 0 & -2 & u_1 - 2u_2 + u_3 \end{array}\right]$$

$\downarrow$

Let $\vec{u} = (1,1,1)$

$c_1 = -\frac{1}{2}(-u_1 - 2u_2 + u_3) = 1$

$c_2 = \frac{1}{2}(-u_1 - 4u_2 + 3u_3) = -1$

$c_3 = -\frac{1}{2}(u_1 - 2u_2 + u_3) = 0$

$c_1(1,2,3) + c_2(0,1,2) + c_3(-1,1,1) = \vec{u}$

$1(1,2,3) + (-1)(0,1,2) + 0(-1,1,1) \stackrel{?}{=} (1,1,1)$

$(1,1,1)$ ✓

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -\frac{1}{2}(-u_1 - 2u_2 + u_3) \\ 0 & 1 & 0 & \frac{1}{2}(-u_1 - 4u_2 + 3u_3) \\ 0 & 0 & 1 & -\frac{1}{2}(u_1 - 2u_2 + u_3) \end{array}\right]$$

Let $\vec{u} = (-1,1,2)$

$c_1 = -\frac{1}{2}$, $c_2 = \frac{3}{2}$, $c_3 = \frac{1}{2}$

$-\frac{1}{2}(1,2,3) + \frac{3}{2}(0,1,2) + \frac{1}{2}(-1,1,1) \stackrel{?}{=} (-1,1,2)$ ✓

DEFINITION OF THE SPAN OF A SET

If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is a set of vectors in a vector space V, then the span of S is the set of all linear combinations of the vectors in S.

$$\text{Span}(S) = \{c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k : c_1, c_2, \ldots, c_k \in \mathbb{R}\}$$

The span of S is denoted by $\text{span}(S)$ or $\text{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$

When $\text{span}(S) = V$, it is said that V is spanned by $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ or that S spans V.

THEOREM 1.6: Span(S) IS A SUBSPACE OF V

If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is a set of a vectors in a vector space V, then $\text{span}(S)$ is a subspace of V. Moreover, $\text{span}(S)$ is the smallest subspace of V that contains S, in the sense that every other subspace of V that contains S must contain $\text{span}(S)$.

Proof:

Let $\vec{u} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k$, $\vec{w} = d_1\vec{v}_1 + d_2\vec{v}_2 + \cdots + d_k\vec{v}_k$ $\in \text{span}(S)$, where c_i, d_i for $i = 1, 2, \ldots, k \in \mathbb{R}$, and $b \in \mathbb{R}$.

span(S) is a nonempty subset of V.

$$\begin{aligned} \vec{u} &= c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k \\ + \quad \vec{w} &= + d_1\vec{v}_1 + d_2\vec{v}_2 + \cdots + d_k\vec{v}_k \\ \hline \vec{u}+\vec{w} &= (c_1+d_1)\vec{v}_1 + (c_2+d_2)\vec{v}_2 + \cdots + (c_k+d_k)\vec{v}_k \in \text{span}(S) \checkmark \end{aligned}$$

$$b\vec{u} = b(c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k)$$
$$b\vec{u} = (b(c_1\vec{v}_1) + b(c_2\vec{v}_2) + \cdots + b(c_k\vec{v}_k))$$
$$b\vec{u} = ((bc_1)\vec{v}_1 + (bc_2)\vec{v}_2 + \cdots + (bc_k)\vec{v}_k) \in \text{span}(S) \checkmark$$

$\therefore$ Span(S) is a subspace of V. //

Example 3: Determine whether the set S spans R^2. If the set does not span R^2, then give a geometric description of the subspace that it does span.

a. $S = \{(1,-1),(2,1)\} = \{\vec{v}_1, \vec{v}_2\}$

Let $\vec{u} = (u_1, u_2)$ be any vector in $\mathbb{R}^2$, u_1 and $u_2 \in \mathbb{R}$.

$$c_1\vec{v}_1 + c_2\vec{v}_2 = \vec{u}$$

$$c_1(1,-1) + c_2(2,1) = (u_1, u_2)$$

$$\boxed{\begin{gathered}S \text{ spans } \mathbb{R}^2 \\ \text{Span}(S) = \mathbb{R}^2\end{gathered}}$$

$$\begin{aligned} c_1 + 2c_2 &= u_1 \\ -c_1 + c_2 &= u_2 \\ \hline 3c_2 &= u_1 + u_2 \\ c_2 &= \tfrac{1}{3}(u_1 + u_2) \end{aligned}$$

$$\begin{aligned} -c_1 + \tfrac{1}{3}(u_1 + u_2) &= u_2 \\ -c_1 + \tfrac{1}{3}u_1 + \tfrac{1}{3}u_2 &= u_2 \\ -c_1 &= -\tfrac{1}{3}u_1 + \tfrac{2}{3}u_2 \\ c_1 &= \tfrac{1}{3}(u_1 - 2u_2) \end{aligned}$$

Check:

$$\tfrac{1}{3}(u_1 - 2u_2)(1,-1) + \tfrac{1}{3}(u_1 + u_2)(2,1) \stackrel{?}{=} (u_1, u_2)$$

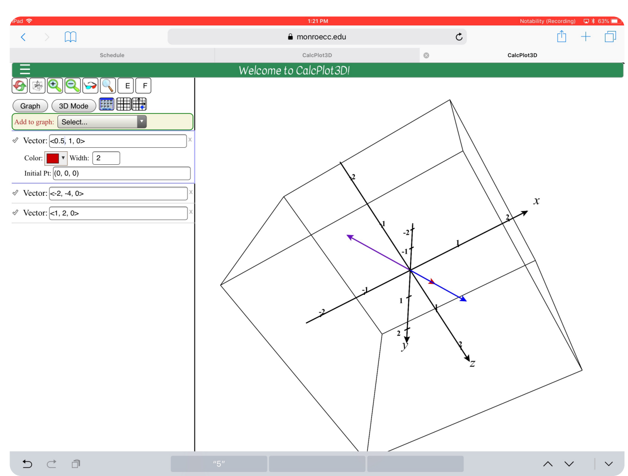
$$(u_1, u_2) = (u_1, u_2) \text{ yep!}$$

b. $S = \left\{(1,2),(-2,-4),\left(\frac{1}{2},1\right)\right\}$

$$c_1(1,2) + c_2(-2,-4) + c_3\left(\tfrac{1}{2},1\right) = (u_1, u_2)$$

$$\begin{aligned} c_1 - 2c_2 + \tfrac{1}{2}c_3 &= u_1 \\ 2c_1 - 4c_2 + c_3 &= u_2 \end{aligned}$$

$$\begin{aligned} -2c_1 + 4c_2 - c_3 &= -2u_1 \\ 2c_1 - 4c_2 + c_3 &= u_2 \\ \hline 0 &= u_2 - 2u_1 \\ u_1 &= \tfrac{1}{2}u_2 \end{aligned}$$



$\boxed{S \text{ does not span } \mathbb{R}^2. \text{ } S \text{ spans the line } y = 2x.}$

c. $S = \{(-1,2),(2,-1),(1,1)\}$
$\vec{v}_1 \quad \vec{v}_2 \quad \vec{v}_3$

Let $\vec{u} = (u_1, u_2)$ be any vector in $\mathbb{R}^2$.

$$c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{u}$$
$$c_1(-1,2) + c_2(2,-1) + c_3(1,1) = (u_1, u_2)$$

$$-c_1 + 2c_2 + c_3 = u_1$$
$$2c_1 - c_2 + c_3 = u_2$$

$$-2c_1 + 4c_2 + 2c_3 = 2u_1$$
$$2c_1 - c_2 + c_3 = u_2$$
$$3c_2 + 3c_3 = 2u_1 + u_2$$
$$c_3 = \tfrac{1}{3}(2u_1 + u_2 - 3c_2)$$

$$-c_1 + 2c_2 + \tfrac{1}{3}(2u_1 + u_2 - 3c_2) = u_1$$
$$-c_1 + 2c_2 + \tfrac{2}{3}u_1 + \tfrac{1}{3}u_2 - c_2 = u_1$$
$$-c_1 + c_2 = \tfrac{1}{3}u_1 - \tfrac{1}{3}u_2$$
$$c_1 = c_2 - \tfrac{1}{3}u_1 + \tfrac{1}{3}u_2$$

See next page

DEFINITION OF LINEAR DEPENDENCE AND LINEAR INDEPENDENCE

A set of vectors $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ in a vector space V is called linearly independent when the vector equation

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k = \vec{0}$$

has only the trivial solution

$$c_1 = c_2 = \cdots = c_k = 0$$

If there are also nontrivial solutions, then S is called linearly dependent.

$$2\left(c_2 - \tfrac{1}{3}u_1 + \tfrac{1}{3}u_2\right) - c_2 + c_3 = u_2$$

$$c_2 - \tfrac{2}{3}u_1 + \tfrac{2}{3}u_2 + c_3 = u_2$$

CRAP!! :)

$$c_2 + \tfrac{1}{3}\left(2u_1 + u_2 - 3c_2\right) = \tfrac{2}{3}u_1$$

redo in matrix:

$$\left[\begin{array}{ccc|c} -1 & 2 & 1 & u_1 \\ 2 & -1 & 1 & u_2 \end{array}\right]$$

$2R1 + R2 \to R2$

$$\left[\begin{array}{ccc|c} -1 & 2 & 1 & u_1 \\ 0 & 3 & 3 & 2u_1 + u_2 \end{array}\right]$$

$$-c_1 + 2c_2 + c_3 = u_1$$
$$c_2 + c_3 = \tfrac{1}{3}(2u_1 + u_2)$$

Let $c_3 = 0$,

$$-c_1 + 2c_2 = u_1$$
$$c_2 = \tfrac{2}{3}u_1 + \tfrac{1}{3}u_2$$

$$-c_1 + 2\left(\tfrac{2}{3}u_1 + \tfrac{1}{3}u_2\right) = u_1$$
$$-c_1 + \tfrac{4}{3}u_1 + \tfrac{2}{3}u_2 = u_1$$
$$c_1 = \tfrac{1}{3}u_1 + \tfrac{2}{3}u_2$$

With $c_3 = 0$:

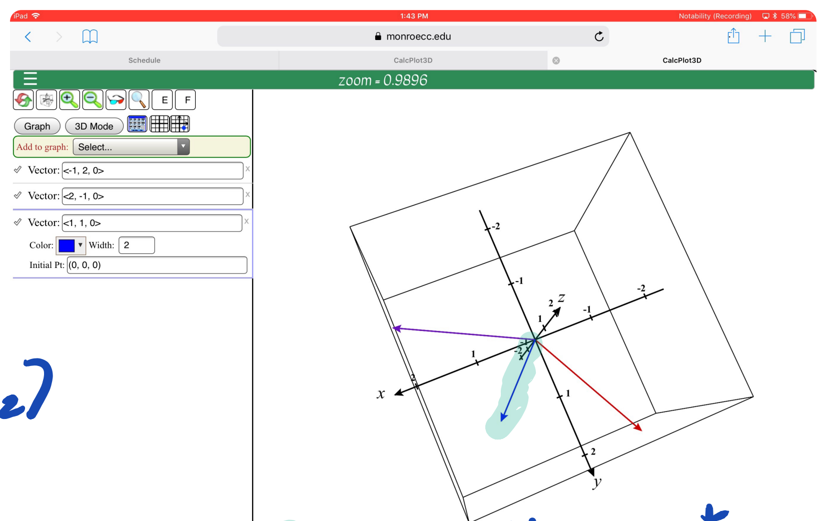
$$c_1(-1,2) + c_2(2,-1) = (u_1, u_2)$$

$$\tfrac{1}{3}(u_1 + 2u_2)(-1,2) + \tfrac{1}{3}(2u_1 + u_2)(2,-1) = (u_1, u_2)$$

Abstract as heck!

Let $\vec{u} = (1,2)$:

$$\left(-\tfrac{5}{3}, \tfrac{10}{3}\right) + \left(\tfrac{8}{3}, -\tfrac{4}{3}\right) = (1,2)$$



zeroed this one out

$\boxed{S \text{ spans } R^2.}$

TESTING FOR LINEAR INDEPENDENCE AND LINEAR DEPENDENCE

Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ be a set of vectors in a vector space V. To determine whether S is linearly independent of linearly dependent, use the following steps.

1. From the vector equation $c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k = \vec{0}$, write a system of linear equations in the variables $c_1, c_2, \ldots,$ and c_k.
2. Use Gaussian elimination to determine whether the system has a unique solution.
3. If the system has only the trivial solution, $c_1 = 0, c_2 = 0, \ldots, c_k = 0$, then the set S is linearly independent. If the system has nontrivial solutions, then S is linearly dependent.

Example 4: Determine whether the set S is linearly independent or linearly dependent.

a. $S = \{(3,-6),(-1,2)\}$
$\quad\vec{v}_1 \quad \vec{v}_2$

$$\left[\begin{array}{cc|c} 3 & -1 & 0 \\ -6 & 2 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 3 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$c_1\vec{v}_1 + c_2\vec{v}_2 = \vec{0}$

$c_1(3,-6) + c_2(-1,2) = (0,0)$

$3c_1 = c_2$

$6c_1 - 2c_2 = 0$

$3c_1 - c_2 = 0$

$-6c_1 + 2c_2 = 0$

$0 = 0$

$c_1(3,-6) + 3c_1(-1,2) = (0,0)$

S is linearly dependent since $\exists$ solutions other than $c_1 = c_2 = 0$.

b. $S = \{(6,2,1),(-1,3,2)\}$
$\quad\vec{v}_1 \quad \vec{v}_2$

$c_1\vec{v}_1 + c_1\vec{v}_2 = \vec{0}$

$c_1(6,2,1) + c_2(-1,3,2) = (0,0,0)$

$6c_1 - c_2 = 0$
$2c_1 + 3c_2 = 0$
$c_1 + 2c_2 = 0$

$$\to \left[\begin{array}{cc|c} 6 & -1 & 0 \\ 2 & 3 & 0 \\ 1 & 2 & 0 \end{array}\right] \text{ or } \left[\begin{array}{cc|c} 6 & -1 & 0 \\ 3 & 5 & 0 \end{array}\right] \text{ or } \begin{bmatrix} 6 & -1 \\ 3 & 5 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right] \qquad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Note: for TI-84 you can't rref a matrix w/more rows than columns.

$$\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \to c_1 = c_2 = 0 \quad \text{so}$$

only solution $\rightarrow$ $0(6,2,1)+0(-1,3,2)=(0,0,0)$
So S is linearly independ.

c. $S=\{(0,0,0,1),(0,0,1,1),(0,1,1,1),(1,1,1,1)\}$
$\vec{v}_1 \quad \vec{v}_2 \quad \vec{v}_3 \quad \vec{v}_3$

$$c_1(0,0,0,1)+c_2(0,0,1,1)+c_3(0,1,1,1)+c_4(1,1,1,1)=(0,0,0,0)$$

$$c_4=0$$
$$c_3+c_4=0$$
$$c_2+c_3+c_4=0$$
$$c_1+c_2+c_3+c_4=0$$

$\boxed{c_1=c_2=c_3=c_4=0 \text{ so } S \text{ is linearly independent}}$

THEOREM 1.7: A PROPERTY OF LINEARLY DEPENDENT SETS

A set $S=\{\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_k\}$, $k\geq 2$, is linearly dependent if and only if at least one of the vectors $\mathbf{v}_j$ can be written as a linear combination of the other vectors in S.

Proof:

1) Suppose S is linearly dependent. Then $\exists$ scalars, not all zero, $\ni$ $c_1\vec{v}_1+c_2\vec{v}_2+\cdots+c_k\vec{v}_k=\vec{0}$. Let $c_1\neq 0$.
Then we have $c_1\vec{v}_1=-c_2\vec{v}_2-c_3\vec{v}_3-\cdots-c_k\vec{v}_k$

$$\vec{v}_1=-\frac{c_2}{c_1}\vec{v}_2-\frac{c_3}{c_1}\vec{v}_3-\cdots-\frac{c_k}{c_1}\vec{v}_k \quad //$$

2) Suppose $\vec{v}_1=c_2\vec{v}_2+c_3\vec{v}_3+\cdots+c_k\vec{v}_k$

$$\vec{0}=-\vec{v}_1+c_2\vec{v}_2+c_3\vec{v}_3+\cdots+c_k\vec{v}_k$$

The coefficient to $\vec{v}_1$ is $-1\neq 0$. $\therefore$ S is linearly dependent. //

THEOREM 1.7: COROLLARY

Two vectors $\mathbf{u}$ and $\mathbf{v}$ in a vector space V are linearly dependent if and only if one is a scalar multiple of the other.

Example 5: Show that the set is linearly dependent by finding a nontrivial linear combination of vectors in the set whose sum is the zero vector. Then express one of the vectors in the set as a linear combination of the other vectors in the set.

$S = \{(2,4),(-1,-2),(0,6)\}$

$c_1(2,4) + c_2(-1,-2) + c_3(0,6) = (0,0)$

$-2(-1,-2) + 0(0,6) = (2,4)$

c_1 c_2

$2c_1 - c_2 = 0$
$4c_1 - 2c_2 + 6c_3 = 0$

$\left[\begin{array}{ccc|c} 2 & -1 & 0 & 0 \\ 4 & -2 & 6 & 0 \end{array}\right] \xrightarrow{-2R1+R2\to R2} \left[\begin{array}{ccc|c} 2 & -1 & 0 & 0 \\ 0 & 0 & 6 & 0 \end{array}\right]$

nontrivial solutions $2c_1 - c_2 = 0 \to c_2 = 2c_1$, $c_3 = 0$

$c_1(2,4) + c_2(-1,-2) = (0,6)$

$2c_1 - c_2 = 0$
$4c_1 - 2c_2 = 6$

$\to \left[\begin{array}{cc|c} 2 & -1 & 0 \\ 4 & -2 & 6 \end{array}\right] \xrightarrow{\frac{1}{2}R2} \left[\begin{array}{cc|c} 2 & -1 & 0 \\ 2 & -1 & 3 \end{array}\right]$

contradiction

$c_1(-1,-2) + c_2(0,6) = (2,4)$

$-c_1 = 2 \to c_1 = -2$

$-2c_1 + 6c_2 = 4$

$\hookrightarrow +4 + 6c_2 = 4 \to c_2 = 0$

$-2(-1,-2) + 0(0,6) = (2,4)$

$c_1 = 1,\ c_2 = 2,\ c_3 = 0$

$1(2,4) + 2(-1,-2) + 0(0,6) = (0,0)$

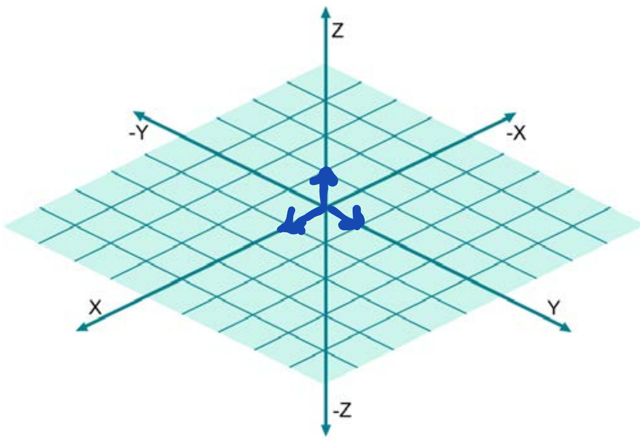
DEFINITION OF BASIS

A set of vectors $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ in a vector space V is called a basis for V when the following conditions are true.

1. S spans V.
2. S is linearly independent.

The Standard Basis for R^3

$S = \{(1,0,0),(0,1,0),(0,0,1)\}$



Example 6: Write the standard basis for the vector space.

a. R^2 $\quad S = \{(1,0),(0,1)\}$

b. R^5 $\quad S = \{(1,0,0,0,0),(0,1,0,0,0),(0,0,1,0,0),$
$(0,0,0,0,1)\}$

c. R^n $\quad S = \{(1,0,0,\ldots,0),(0,1,0,0,\ldots,0),\ldots,(0,0,0,\ldots,0,1,0),$
$(0,0,\ldots,0,0,1)\}$

n vectors

Example 7: Determine whether S is a basis for the indicated vector space.

$S = \{(2,1,0),(0,-1,1)\}$ for R^3

Let $u = (u_1, u_2, u_3)$ be any vector in R^3.

$c_1(2,1,0) + c_2(0,-1,1) = (u_1,u_2,u_3)$

$$\left.\begin{aligned} 2c_1 &= u_1 \to c_1 = \tfrac{1}{2}u_1 \\ c_1 - c_2 &= u_2 \to c_2 = c_1 + u_2 \\ c_2 &= u_3 \end{aligned}\right\}$$

$\frac{1}{2}u_1(2,1,0) + u_3(0,-1,1) = (u_1,u_2,u_3)$

Let's check the system:

Let $u = (1,2,3)$

$\frac{1}{2}\cdot 1(2,1,0) + 3(0,-1,1) \overset{?}{=} (1,2,3)$

$(1,\frac{1}{2},0) + (0,-3,3) \overset{?}{=} (1,2,3)$ → $(1,-\frac{5}{2},3) \neq (1,2,3)$

S is not a basis for R^3 since S doesn't span R^3.

THEOREM 1.8: UNIQUENESS OF BASIS REPRESENTATION

If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis for a vector space V, then every vector in V can be written in one and only one way as a linear combination of vectors in S.

Proof:

Since S is a basis for V, S is linearly independent

$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n = \vec{0}$ ~~implies that~~

~~$c_1 = c_2 = \cdots = c_n = 0$. So the only way to write~~

~~combination~~

$c_1\vec{v}_1 = -c_2\vec{v}_2 - c_3\vec{v}_3 - \cdots - c_n\vec{v}_n$. Since S is lin. ind. we know that $c_1 = 0$, so we can't mult. both sides by $\frac{1}{c_1}$.

Evil plan

- Basis means S spans V and S is lin. ind.

Proof: Since S is a basis for V, S spans V and S is linearly independent.

Let $\vec{u} = c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_n\vec{v}_n$ and suppose $\vec{u}$ can also be written as $\vec{u} = b_1\vec{v}_1 + b_2\vec{v}_2 + \dots + b_n\vec{v}_n$.

$$\begin{array}{rl} \vec{u} &= c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_n\vec{v}_n \\ -\ \vec{u} &= -(b_1\vec{v}_1 + b_2\vec{v}_2 + \dots + b_n\vec{v}_n) \\ \hline \vec{0} &= (c_1\vec{v}_1 - b_1\vec{v}_1) + (c_2\vec{v}_2 - b_2\vec{v}_2) + \dots + (c_n\vec{v}_n - b_n\vec{v}_n) \end{array}$$

$$\vec{0} = (c_1 - b_1)\vec{v}_1 + (c_2 - b_2)\vec{v}_2 + \dots + (c_n - b_n)\vec{v}_n$$

Since is linearly independent,

$$c_1 - b_1 = 0,\ c_2 - b_2 = 0,\ \dots,\ c_n - b_n = 0$$

$$c_1 = b_1,\ c_2 = b_2,\ \dots,\ c_n = b_n$$

Thus the basis representation is unique. //

THEOREM 1.9: BASES AND LINEAR DEPENDENCE

If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis for a vector space V, then every set containing more than n vectors in V is linearly dependent.

THEOREM 1.10: NUMBER OF VECTORS IN A BASIS

If a vector space V has one basis with n vectors, then every basis for V has n vectors.

Proof: Let $S_1 = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ be a basis for V, and suppose $S_2 = \{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_m\}$ also be a basis for V.

Since S_1 is linearly independend and S_2 spans V, $n \le m$ [Thm. 1.9]. Similarly, since S_2 is linearly independent and S_1 spans V, $m \le n$ [Thm. 1.9]. Hence, $m = n$. //

DEFINITION OF DIMENSION OF A VECTOR SPACE

If a vector space V has a basis consisting of n vectors, then the number n is called the dimension of V, denoted by dim(V). When V consists of the zero vector alone, the dimension of V is defined as 0.

Example 8: Determine the dimension of the vector space.

a. R^2

$\dim(R^2) = 2$

b. R^5

$\dim(R^5) = 5$

c. R^n

$\dim(R^n) = n$

THEOREM 1.11: BASIS TESTS IN AN n-DIMENSIONAL SPACE

Let V be a vector space of dimension n.

1. If $S=\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is a linearly independent set of vectors in V, then S is a basis for V.
2. If $S=\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ spans V, then S is a basis for V.

Example 9: Determine whether S is a basis for the indicated vector space.

$S=\{(1,2),(1,-1)\}$ for R^2. $\dim(R^2)=2$

$$c_1(1,2)+c_2(1,-1)=(0,0)$$

$$c_1+c_2=0$$
$$2c_1-c_2=0$$
$$3c_1=0$$
$$c_1=0$$
$$c_2=0$$

$\boxed{c_1=c_2=0 \rightarrow S \text{ is linearly independ. and } S \text{ has 2 vectors and } 2=\dim(R^2) \text{ so } S \text{ is a basis for } R^2.}$

2.1 Matrix Operations

Learning Objectives

1. Determine whether two matrices are equal
2. Add and subtract matrices, and multiply a matrix by a scalar
3. Multiply two matrices
4. Use matrices to solve a system of equations
5. Partition a matrix and write a linear combination of column vectors

Matrices can be thought of as adjoined column vectors. They are represented in the following ways:

1. Capital letter A, B, C
2. Representative element $A = [a_{ij}]$
3. Rectangular array

$$B = \begin{bmatrix} b_{11} & b_{12} & b_{13} & \cdots & b_{1n} \\ b_{21} & b_{22} & b_{23} & \cdots & b_{2n} \\ b_{31} & b_{32} & b_{33} & \cdots & b_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ b_{m1} & b_{m2} & b_{m3} & \cdots & b_{mn} \end{bmatrix}$$

DEFINITION OF EQUALITY OF MATRICES

Two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are equal when they have the same size $m \times n$ and $a_{ij} = b_{ij}$ for $1 \le i \le m$ and $1 \le j \le n$.

Example 1: Are matrices A and B equal? Please explain.

$A = \begin{bmatrix} 1 & -1 & 3 & 8 \end{bmatrix}$ 1×4

$B = \begin{bmatrix} 1 \\ -1 \\ 3 \\ 8 \end{bmatrix}$ 4×1

No → not the same size!

Example 2: Find x and y.

$$\begin{bmatrix} 2x-1 & 4 \\ 3 & y^3 \end{bmatrix} = \begin{bmatrix} -5 & 4 \\ 3 & \frac{1}{8} \end{bmatrix}$$

$2x-1=-5$

$\boxed{x=-2}$

$(y^3)^{1/3} = \left(\frac{1}{8}\right)^{1/3}$

$\boxed{y=\frac{1}{2}}$

A matrix that has only one column is called a column matrix or column vector. A matrix that has only one row is called a row matrix or row vector. As we learned earlier, boldface lowercase letters often designate row matrix and column matrix.

$$\vec{a}_1 = \begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix} \qquad A = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 \end{bmatrix}$$

$$\vec{a}_2 = \begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix} \qquad A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

DEFINITION OF MATRIX ADDITION

If $A=[a_{ij}]$ and $B=[b_{ij}]$ are matrices of size $m \times n$, then their sum is the $m \times n$ matrix given by

$A+B=[a_{ij}+b_{ij}]$

The sum of two matrices of different sizes is undefined.

DEFINITION OF SCALAR MULTIPLICATION

If $A=[a_{ij}]$ is an $m \times n$ matrix and c is a scalar, then the scalar multiple of A by c is the $m \times n$ matrix given by

$cA=[ca_{ij}]$

Note: You can use $-A$ to represent the scalar product $(-1)A$. If A and B are of the same size, then $A-B$ represents the sum of A and $-B$.

Example 3: Find the following for the matrices

$$A=\begin{bmatrix} 1 & -3 & 6 \\ 2 & 0 & 2 \\ -2 & 8 & -1 \end{bmatrix} \text{ and } B=\begin{bmatrix} 5 & 2 & 7 \\ -1 & 9 & -4 \\ -3 & 0 & 1 \end{bmatrix}$$

a. $A+B$

$$=\begin{bmatrix} 6 & -1 & 13 \\ 1 & 9 & -2 \\ -5 & 8 & 0 \end{bmatrix}$$

b. $2A-B$

$$=\begin{bmatrix} 2 & -6 & 12 \\ 4 & 0 & 4 \\ -4 & 16 & -2 \end{bmatrix} + \begin{bmatrix} -5 & -2 & -7 \\ +1 & -9 & +4 \\ +3 & -0 & -1 \end{bmatrix} \rightarrow = \boxed{\begin{bmatrix} -3 & -8 & 5 \\ 5 & -9 & 8 \\ -1 & 16 & -3 \end{bmatrix}}$$

DEFINITION OF MATRIX MULTIPLICATION

If $A=\left[a_{ij}\right]$ is an $m\times n$ matrix and $B=\left[b_{ij}\right]$ is an $n\times p$ matrix, then the product AB is an $m\times p$ matrix.

$$AB = C = [c_{ij}]$$

where

$$c_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$$

To find an entry in the ith row and the jth column of the product AB, multiply the <u>entries</u> in the <u>ith</u> row of A by the corresponding entries in the <u>jth</u> column of B and then <u>Sum</u> the results.

Example 4: Find the product AB, where

$$A = \begin{bmatrix} 15 & 0 \\ 4 & 5 \\ -3 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} -12 & 7 & 5 & -1 \\ -13 & 1 & 2 & 11 \end{bmatrix}$$

A times B

3×2 2×4

resulting size is 3×4

$$\begin{bmatrix} 15 & 0 \\ 4 & 5 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} -12 & 7 & 5 & -1 \\ -13 & 1 & 2 & 11 \end{bmatrix}$$

$$= \begin{bmatrix} 15(-12)+0(-13) & 15(7)+0(1) & 15(5)+0(2) & 15(-1)+0(11) \\ 4(-12)+5(-13) & 4(7)+5(1) & 4(5)+5(2) & 4(-1)+5(11) \\ -3(-12)+1(-13) & -3(7)+1(5) & -3(5)+1(2) & -3(-1)+1(11) \end{bmatrix}$$

$$= \boxed{\begin{bmatrix} -180 & 105 & 75 & -15 \\ -113 & 33 & 30 & 51 \\ 23 & -20 & -13 & 14 \end{bmatrix}}$$

Example 5: Consider the matrices A and B.

$A = \begin{bmatrix} -1 & 3 \\ 11 & 13 \end{bmatrix}$ and $B = \begin{bmatrix} -4 & 4 \\ 6 & 13 \end{bmatrix}$

2×2 $\quad$ 2×2

a. Find $A+B$

b. Find $B+A$

$$A+B = \begin{bmatrix} -1+(-4) & 3+4 \\ 11+6 & 13+13 \end{bmatrix} = \begin{bmatrix} -5 & 7 \\ 17 & 26 \end{bmatrix} = \begin{bmatrix} -4+(-1) & 4+3 \\ 6+11 & 13+13 \end{bmatrix} = B+A$$

c. Find AB

$$\begin{bmatrix} -1 & 3 \\ 11 & 13 \end{bmatrix}\begin{bmatrix} -4 & 4 \\ 6 & 13 \end{bmatrix} = \begin{bmatrix} (-1)(-4)+(3)(6) & (-1)(4)+(3)(13) \\ (11)(-4)+(13)(6) & (11)(4)+(13)(13) \end{bmatrix}$$

$$= \boxed{\begin{bmatrix} 22 & 35 \\ 34 & 213 \end{bmatrix}}$$

d. Find BA

$$\begin{bmatrix} -4 & 4 \\ 6 & 13 \end{bmatrix}\begin{bmatrix} -1 & 3 \\ 11 & 13 \end{bmatrix} = \begin{bmatrix} (-4)(-1)+(4)(11) & (-4)(3)+(4)(13) \\ (6)(-1)+(13)(11) & (6)(3)+(13)(13) \end{bmatrix}$$

$$= \boxed{\begin{bmatrix} 48 & 40 \\ 137 & 187 \end{bmatrix}}$$

Is matrix addition commutative?

It looks like it might be :)

Is matrix multiplication commutative?

$\boxed{\text{NO}}$

$$\begin{array}{ccc} \vec{a}_1 & \vec{a}_2 & \vec{a}_3 \end{array} \quad A \qquad \vec{x} \qquad \vec{b}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$3 \times 3 \qquad 3 \times 1$

$$\vec{a}_1 x_1 = \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix} x_1 = \begin{bmatrix} a_{11}x_1 \\ a_{21}x_1 \\ a_{31}x_1 \end{bmatrix}$$

$=$

SYSTEMS OF LINEAR EQUATIONS

The system

$$\begin{bmatrix} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

can be written as

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

or equivalently, $A\vec{x} = \vec{b}$

Example 6: Write the system of equations in the form $A\mathbf{x} = \mathbf{b}$ and solve this matrix equation for $\mathbf{x}$.

$$\begin{aligned} 2x_1 + 3x_2 &= 5 \\ x_1 + 4x_2 &= 10 \end{aligned}$$

$$A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}, \quad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 5 \\ 10 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \end{bmatrix} \longrightarrow \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \end{bmatrix} \checkmark$$

$$\left[\begin{array}{cc|c} 2 & 3 & 5 \\ 1 & 4 & 10 \end{array}\right]$$

$-2R2 + R1 \to R2$

$$\left[\begin{array}{cc|c} 2 & 3 & 5 \\ 0 & -5 & -15 \end{array}\right] \to \left[\begin{array}{cc|c} 2 & 3 & 5 \\ 0 & 1 & 3 \end{array}\right]$$

$-3R2 + R1 \to R1$

$$\left[\begin{array}{cc|c} 2 & 0 & -4 \\ 0 & 1 & 3 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & 0 & -2 \\ 0 & 1 & 3 \end{array}\right] \to x_1 = -2, \; x_2 = 3$$

PARTITIONED MATRICES

$$A = \begin{array}{c} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \\ \begin{array}{cccc} \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n \end{array} \end{array} \qquad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

LINEAR COMBINATIONS (MATRICES)

The matrix product $A\mathbf{x}$ is a linear combination of the column vectors $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \ldots, \mathbf{a}_n$ that form the coefficient matrix A.

$$A\vec{x} = x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

$$A\vec{x} = x_1\vec{a}_1 + x_2\vec{a}_2 + \cdots + x_n\vec{a}_n$$

The system $A\mathbf{x} = \mathbf{b}$ is consistent if and only if $\mathbf{b}$ can be expressed as such a linear combination, where the coefficients of the linear combination are a solution of the system.

Example 7: Write the column matrix $\mathbf{b}$ as a linear combination of the columns of A

$$A = \begin{array}{c} \begin{array}{cc} \vec{a}_1 & \vec{a}_2 \end{array} \\ \begin{bmatrix} -1 & 3 \\ 16 & 1 \end{bmatrix} \end{array}, \quad \mathbf{b} = \begin{bmatrix} -7 \\ 63 \end{bmatrix}, \quad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$x_1\vec{a}_1 + x_2\vec{a}_2 = \vec{b}$$

$$x_1 \begin{bmatrix} -1 \\ 16 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} -7 \\ 63 \end{bmatrix}$$

$$\boxed{4 \begin{bmatrix} -1 \\ 16 \end{bmatrix} + (-1) \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} -7 \\ 63 \end{bmatrix}}$$

$$\begin{bmatrix} -1x_1 \\ 16x_1 \end{bmatrix} + \begin{bmatrix} 3x_2 \\ 1x_2 \end{bmatrix} = \begin{bmatrix} -7 \\ 63 \end{bmatrix}$$

$$\begin{bmatrix} -x_1 + 3x_2 \\ 16x_1 + x_2 \end{bmatrix} = \begin{bmatrix} -7 \\ 63 \end{bmatrix}$$

$$-x_1 + 3x_2 = -7$$
$$16x_1 + x_2 = 63$$

$$x_1 = 4, \ x_2 = -1$$

Example 8: Find the products AB and BA for the diagonal matrices.

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & 5 \end{bmatrix} \qquad B = \begin{bmatrix} -7 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 12 \end{bmatrix}$$

$$AB = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} -7 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 12 \end{bmatrix}$$

$$= \begin{bmatrix} 3(-7)+0(0)+0(0) & 3(0)+0(4)+0(0) & 3(0)+0(0)+0(12) \\ 0(-7)+(-5)(0)+(0)(0) & 0(0)+(-5)(4)+(0)(0) & 0(0)+(-5)(0)+0(12) \\ 0(-7)+0(0)+5(0) & (0)(0)+0(4)+(5)(0) & 0(0)+(0)(0)+5(12) \end{bmatrix}$$

$$= \boxed{\begin{bmatrix} -21 & 0 & 0 \\ 0 & -20 & 0 \\ 0 & 0 & 60 \end{bmatrix}}$$

Example 9: Use the given partitions of A and B to compute AB.

$$A = \left[\begin{array}{c|c} 2 & 1 \\ -1 & 0 \\ \hline 3 & 1 \end{array}\right], \text{ and } B = \left[\begin{array}{cc} 3 & 0 \\ \hline 2 & 1 \end{array}\right]$$

3×2 $\quad$ 2×2

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad B = \begin{bmatrix} B_{11} \\ B_{21} \end{bmatrix}$$

$$A_{11} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \quad A_{12} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$A_{21} = [3] \quad A_{22} = [1]$$

$$B_{11} = [3 \ 0] \quad B_{21} = [2 \ 1]$$

$$AB = \begin{bmatrix} A_{11}B_{11} + A_{12}B_{21} \\ A_{21}B_{11} + A_{22}B_{21} \end{bmatrix}$$

$$\begin{bmatrix} 2 \\ -1 \end{bmatrix}[3 \ 0] = \begin{bmatrix} 6 & 0 \\ -3 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}[2 \ 1] = \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}$$

$$[3][3 \ 0] = [9 \ 0]$$

$$[1][2 \ 1] = [2 \ 1]$$

$$= \boxed{\begin{bmatrix} 8 & 1 \\ -3 & 0 \\ 11 & 1 \end{bmatrix}}$$

2.2: Properties of Matrix Operations

Learning Objectives

1. Use the properties of matrix addition, scalar multiplication, and zero matrices
2. Use the properties of matrix multiplication and the identity matrix
3. Find the transpose of a matrix
4. Use Stochastic matrices for applications

THEOREM 2.1: PROPERTIES OF MATRIX ADDITION AND SCALAR MULTIPLICATION

If A, B, and C are $m \times n$ matrices, and c and d are scalars, then the following properties are true.

Let $A=[a_{ij}]$, $B=[b_{ij}]$, $C=[c_{ij}] \ni a_{ij}, b_{ij}$, and c_{ij}, $1 \le i \le m$, $1 \le j \le n$, $\in \mathbb{R}$.

1. $A+B =$ $B+A$ Commutative property of addition

Proof:

$A+B = [a_{ij}]+[b_{ij}]$

$= [a_{ij}+b_{ij}]$ defn matrix(+)

$= [b_{ij}+a_{ij}]$ $\mathbb{R}$ is comm.(+)

$= [b_{ij}]+[a_{ij}]$ defn matrix(+)

$= B+A$ //

2. $A+(B+C) =$ $(A+B)+C$ Associative property of addition

3. $(cd)A =$ $c(dA)$ Associative property of multiplication

$(cd)A = (cd)[a_{ij}]$

$= [(cd)a_{ij}]$ defn matrix scalar mult

$= [c(da_{ij})]$ $\mathbb{R}$ is assoc (×)

$= c[da_{ij}]$ defn matrix scal. mult

$= c(d[a_{ij}])$

$= c(dA)$ //

4. $1A =$ A Scalar Multiplicative Identity

5. $c(A+B) =$ $cA+cB$ Distributive property

Proof:

$c(A+B) = c([a_{ij}]+[b_{ij}])$

$= c[a_{ij}+b_{ij}]$ defn matrix (+)

$= [c(a_{ij}+b_{ij})]$ defn matrix scal. mult

$= [ca_{ij}+cb_{ij}]$ $\mathbb{R}$ is distributive

$= [ca_{ij}]+[cb_{ij}]$ defn matrix (+)

$= c[a_{ij}]+c[b_{ij}]$ defn matrix scalar mult

$= cA+cB$ //

6. $(c+d)A =$ $cA+dA$ Distributive property

Example 1: For the matrices below, $c=-2$, and $d=5$,

$$A=\begin{bmatrix}-3 & 5\\ 3 & 4\\ 4 & 8\end{bmatrix} \qquad B=\begin{bmatrix}1 & 1\\ 2 & 7\\ 6 & 9\end{bmatrix} \qquad C=\begin{bmatrix}-7 & 1\\ -2 & 3\\ 11 & 2\end{bmatrix}$$

a. $c(A+C)$ $= -2\begin{bmatrix}-10 & 6\\ 1 & 7\\ 15 & 10\end{bmatrix}$

$= \boxed{\begin{bmatrix}+20 & -12\\ -2 & -14\\ -30 & -20\end{bmatrix}}$

b. cdB $= -10\begin{bmatrix}1 & 1\\ 2 & 7\\ 6 & 9\end{bmatrix}$

$= \boxed{\begin{bmatrix}-10 & -10\\ -20 & -70\\ -60 & -90\end{bmatrix}}$

c. $cA-(B+C)$ $= \boxed{\begin{bmatrix}12 & -12\\ -6 & -18\\ -25 & -27\end{bmatrix}}$

THEOREM 2.2: PROPERTIES OF ZERO MATRICES

If A is an $m \times n$ matrix, and c is a scalar, then the following properties are true.

1. $A + O_{mn} =$ $\underline{A}$ additive identity
2. $A + (-A) =$ $\underline{O}$ additive inverse
3. If $cA = O_{mn}$, then $\underline{c = 0 \text{ or } A = O_{mn}}$.

Example 2: Solve for X in the equation, given

$$A = \begin{bmatrix} -2 & -1 \\ 1 & 0 \\ 3 & -4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 3 \\ 2 & 0 \\ -4 & -1 \end{bmatrix}$$

a. $X = 3A - 2B$

$$X = \begin{bmatrix} -6 & -3 \\ 3 & 0 \\ 9 & -12 \end{bmatrix} + \begin{bmatrix} 0 & -6 \\ -4 & 0 \\ 8 & 2 \end{bmatrix}$$

$$\boxed{X = \begin{bmatrix} -6 & -9 \\ -1 & 0 \\ 17 & -10 \end{bmatrix}}$$

b. $-\tfrac{1}{2}(2A + 4B) = -\tfrac{1}{2}(-2X)$

$$-A - 2B = X$$

$$\begin{bmatrix} 2 & 1 \\ -1 & 0 \\ -3 & 4 \end{bmatrix} + \begin{bmatrix} 0 & -6 \\ -4 & 0 \\ 8 & 2 \end{bmatrix} = X$$

$$\boxed{\begin{bmatrix} 2 & -5 \\ -5 & 0 \\ 5 & 6 \end{bmatrix} = X}$$

$x(z+y)$
$= zx + xy \Rightarrow$ with matrices you can't ever assume comm.

THEOREM 2.3: PROPERTIES OF MATRIX MULTIPLICATION

If A, B, and C are matrices (**with sizes such that the given matrix products are defined**), and c is a scalar, then the following properties are true.

1. $A(BC) =$ $(AB)C$ Associative property of multiplication
2. $A(B+C) =$ $AB + AB$ Distributive property of multiplication
3. $(A+B)C =$ $AC + BC$ Distributive property of multiplication
4. $c(AB) = (cA)B =$ $A(cB)$

Example 3: Show that $AC = BC$, even though $A \neq B$.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 4 \\ 3 & -2 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 4 & -6 & 3 \\ 5 & 4 & 4 \\ -1 & 0 & 1 \end{bmatrix} \qquad C = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 4 & -2 & 1 \end{bmatrix}$$

$$AC = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 4 \\ 3 & -2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 4 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 12 & -6 & 3 \\ 16 & -8 & 4 \\ 4 & -2 & 1 \end{bmatrix}$$

$$BC = \begin{bmatrix} 4 & -6 & 3 \\ 5 & 4 & 4 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 4 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 12 & -6 & 3 \\ 16 & -8 & 4 \\ 4 & -2 & 1 \end{bmatrix}$$

$AC = BC$

Example 4: Show that $AB = \mathbf{0}$, even though $A \neq \mathbf{0}$ and $B \neq \mathbf{0}$.

$$A = \begin{bmatrix} 2 & 4 \\ 2 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & -2 \\ -\frac{1}{2} & 1 \end{bmatrix}$$

$$AB = \begin{bmatrix} 2 & 4 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ -\frac{1}{2} & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$ wow!! ☺

THEOREM 2.4: PROPERTIES OF THE IDENTITY MATRIX

If A is an $m \times n$ matrix, then the following properties are true.

1. $AI_n =$ <u>A</u>
2. $I_m A =$ <u>A</u>

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$I_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & 0 \cdots & 0 \\ \vdots & 0 & \ddots & \vdots \\ 0 & 0 & \cdots 0 & 1 \end{bmatrix}$$

THEOREM 2.5: NUMBER OF SOLUTIONS OF A LINEAR SYSTEM

For a system of linear equations, precisely one of the following is true.

1. The system has exactly <u>one</u> solution.
2. The system has <u>infinitely</u> many solutions.
3. The system has <u>no</u> solution.

In Octave: A' and trans(A)

THE TRANSPOSE OF A MATRIX

The transpose of a matrix is denoted A^T and is formed by writing its columns as rows.

Example 5: Find the transpose of the matrix.

a. $A=\begin{bmatrix}1 & -1\\ 2 & 9\\ 4 & 10\end{bmatrix}$ 3×2

$A^T = \begin{bmatrix}1 & 2 & 4\\ -1 & 9 & 10\end{bmatrix}$ 2×3

b. $A=\begin{bmatrix}6 & -7 & 19\\ -7 & 0 & 23\\ 19 & 23 & -32\end{bmatrix}$ 3×3

$A^T = \begin{bmatrix}6 & -7 & 19\\ -7 & 0 & 23\\ 19 & 23 & -32\end{bmatrix}$ 3×3

If $A=A^T$, A is symmetric

THEOREM 2.6: PROPERTIES OF TRANSPOSES

If A and B are matrices (with sizes such that the given matrix operations are defined), and c is a scalar, then the following properties are true. Let $A=[a_{ij}]$, $B=[b_{ij}]$ $\ni$ $a_{ij}, b_{ij} \in \mathbb{R}$

1. $(A^T)^T =$ A Transpose of a transpose

Proof:

$(A^T)^T = ([a_{ij}]^T)^T$
$= [a_{ji}]^T$
$= [a_{ij}]$
$= A$ //

2. $(A+B)^T =$ A^T+B^T Transpose of a sum

Proof:

$(A+B)^T = ([a_{ij}]+[b_{ij}])^T$

$= [a_{ij}+b_{ij}]^T$ defn of matrix(+)

$= [a_{ji}+b_{ji}] \rightarrow = [a_{ji}]+[b_{ji}] \rightarrow = A^T+B^T$ //

3. $(cA)^T =$ cA^T Transpose of a scalar multiple

4. $(AB)^T =$ B^TA^T Transpose of a product

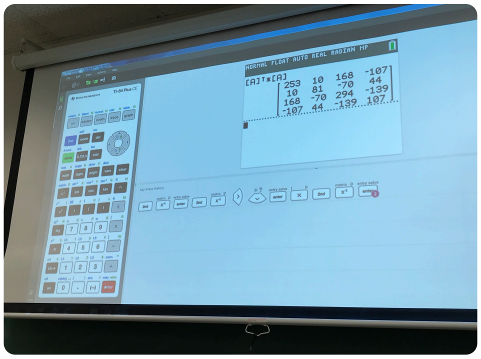
A is $m\times p$ A^T is $n\times m$

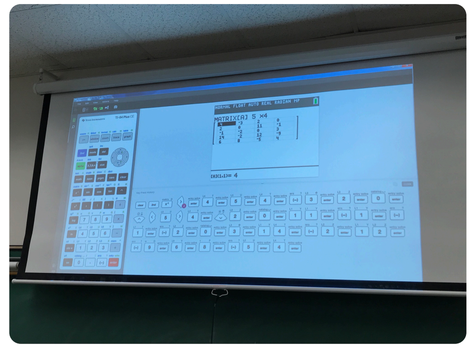
B is $n\times p$ B^T is $p\times n$

$\underset{p\times n}{B^T}\,\underset{n\times m}{A^T}$ is defined

Example 6: Find a) $A^T A$ and b) AA^T. Show that each of these products is symmetric.

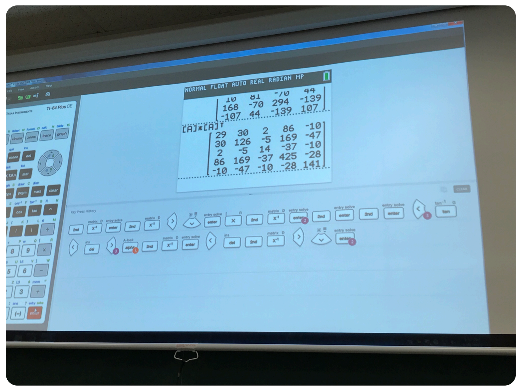
$$A = \begin{bmatrix} 4 & -3 & 2 & 0 \\ 2 & 0 & 11 & -1 \\ -1 & -2 & 0 & 3 \\ 14 & -2 & 12 & -9 \\ 6 & 8 & -5 & 4 \end{bmatrix}$$





A is 5×4

A^T is 4×5



Example 7: A square matrix is called skew-symmetric when $A^T = -A$. Prove that if A and B are skew-symmetric matrices, then $A+B$ is skew-symmetric.

Evil Plan

$(A+B)^T = -(A+B)$

$$\begin{aligned}(A+B)^T &= A^T + B^T \\ &= -A + (-B) \quad \text{[A and B are skew-symmetric]} \\ &= -1(A+B) \\ &= -(A+B) \quad //\end{aligned}$$

STOCHASTIC MATRICES

Many types of applications involve a finite set of __States__ __$\{S_1, S_2, \dots, S_n\}$__ of a given population. The __probability__ that a member of a population will change from the __jth__ state to the __ith__ state is represented by a number __P_{ij}__, where __$0 \le P_{ij} \le 1$__. A probability of __0__ means that the member is certain __not__ to change from the *j*th state to the *i*th state whereas a probability of __1__ means that the member is __certain__ to change from the *j*th state to the *i*th state.

$$P = \begin{bmatrix} P_{11} & P_{12} & \cdots & P_{1n} \\ P_{21} & P_{22} & \cdots & P_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ P_{n1} & P_{n2} & \cdots & P_{nn} \end{bmatrix}$$

***P* is called the __matrix__ of __transition__ probabilities.** At each transition, each member in a given state must either stay in that state or change to another state. Therefore, the sum of the entries in any __column__ is __1__. This type of matrix is called __Stochastic__. An __n x n__ matrix *P* is a **stochastic matrix** when each entry is a number between __0__ and __1__ inclusive.

Example 8: Determine whether the matrix is stochastic.

$$A = \begin{bmatrix} 0.35 & 0.2 \\ 0.65 & 0.75 \end{bmatrix}$$

1 0.95

not stochastic

$$B = \begin{bmatrix} \frac{1}{8} & \frac{3}{5} & \frac{1}{12} \\ \frac{1}{2} & \frac{1}{10} & \frac{1}{3} \\ \frac{3}{8} & \frac{3}{10} & \frac{7}{12} \end{bmatrix}$$

1 1 1

B is a stochastic matrix

Example 9: A medical researcher is studying the spread of a virus in a population of 1000 laboratory mice. During any week, there is an 80% probability that an infected mouse will overcome the virus, and during the same week, there is a 10% probability that a noninfected mouse will become infected. One hundred mice are currently infected with the virus. How many will be infected (a) next week and (b) in two weeks?

$$P = \begin{array}{cc|c} I & NI & \\ \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} & & \begin{array}{c} I \\ NI \end{array} \end{array} \qquad X_0 = \begin{bmatrix} 100 \\ 900 \end{bmatrix} \begin{array}{c} I \\ NI \end{array}$$

$$P = \begin{bmatrix} 0.2 & 0.1 \\ 0.8 & 0.9 \end{bmatrix}$$

a) $$PX_0 = \begin{bmatrix} 0.2 & 0.1 \\ 0.8 & 0.9 \end{bmatrix} \begin{bmatrix} 100 \\ 900 \end{bmatrix} = \begin{bmatrix} 110 \\ 890 \end{bmatrix} \begin{array}{c} I \\ NI \end{array} = X_1$$

$\boxed{\text{Next week, 110 mice will be infected.}}$

b) $$P[PX_0] = PX_1 \rightarrow P^2X_0$$

$$= \begin{bmatrix} 0.2 & 0.1 \\ 0.8 & 0.9 \end{bmatrix} \begin{bmatrix} 110 \\ 890 \end{bmatrix}$$

$$= \begin{bmatrix} 111 \\ 889 \end{bmatrix}$$

$\boxed{\text{In 2 weeks, 111 mice will be infected.}}$

$$= X_2$$

c) In 10 weeks, we'd just use $P^{10}X_0$

```
octave:2> P = [0.2 0.1; 0.8 0.9]
P =

   0.20000   0.10000
   0.80000   0.90000

octave:3> X0 = [100; 900]
X0 =

   100
   900

octave:4> P*X0
ans =

   110
   890

octave:5> P^2*X0
ans =

   111.00
   889.00

octave:6> P^10*X0
ans =

   111.11
   888.8
```

Example 10: It has been claimed that the best predictor of today's weather is yesterday's weather. Suppose that in San Diego, if it rained yesterday, then there is a 20% chance of rain today, and if it did not rain yesterday, then there is a 90% chance of no rain today.

a. Find the transition matrix describing the rain probabilities.

$$P = \begin{array}{c|cc|c} & R & NR & \\ & .2 & .1 & R \\ & .8 & .9 & NR \end{array} \qquad X_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{array}{c} R \\ NR \end{array}$$

b. If it rained Sunday, what is the chance of rain on Tuesday?

$$\left(\begin{bmatrix} .2 & .1 \\ .8 & .9 \end{bmatrix}\right)^2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} .12 \\ .88 \end{bmatrix}$$

On Tuesday, there's a 12% chance of rain.

c. If it did not rain on Wednesday, what is the chance of rain on Saturday?

$$\left(\begin{bmatrix} 0.2 & 0.1 \\ 0.8 & 0.9 \end{bmatrix}\right)^3 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.11 \\ 0.89 \end{bmatrix}$$

On Saturday, there's an 11% chance of rain.

d. If the probability of rain today is 30%, what is the chance of rain tomorrow?

$$\begin{bmatrix} 0.2 & 0.1 \\ 0.8 & 0.9 \end{bmatrix} \begin{bmatrix} 0.3 \\ 0.7 \end{bmatrix} = \begin{bmatrix} .13 \\ .87 \end{bmatrix}$$

There would be a 13% chance of rain tomorrow.

2.3: The Inverse of a Matrix

Learning Objectives

1. Find the inverse of a matrix (if it exists)
2. Use properties of inverse matrices
3. Use an inverse matrix to solve a system of linear equations
4. Encode and decode messages
5. Elementary Matrices
6. *LU*-Factorization

DEFINITION OF THE INVERSE OF A MATRIX

An $n \times n$ matrix A is invertible or nonsingular when there exists an $n \times n$ matrix B such that

$$AB = BA = I_n$$

where I_n is the identity matrix of order n. The matrix B is called the (multiplicative) inverse of A. A matrix that does not have an inverse is called noninvertible or singular.

***Nonsquare matrices do not have inverses.**

Example 1: For the matrices below, show that B is the inverse of A.

$$A = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

$$AB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \checkmark \qquad BA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \checkmark$$

THEOREM 2.7: UNIQUENESS OF AN INVERSE

If A is an invertible matrix, then its inverse is unique. The inverse of A is denoted A^{-1}.

Proof: Since A is invertible we know $\exists$ a $B \ni AB = I = BA$. Suppose $\exists$ a $C \ni AC = I = CA$.

$C(AB) = CI$

$(CA)B = C$

$IB = C$

$B = C$.

$\therefore$ The inverse of A is unique. //

FINDING THE INVERSE OF A MATRIX BY GAUSS-JORDAN ELIMINATION

Let A be a square matrix of order n.

1. Write the $n \times 2n$ matrix that consists of the given matrix A on the left and the $n \times n$ identity matrix I_n on the right to obtain $[A \; I_n]$. This process is called adjoining matrix I to matrix A.
2. If possible, row reduce A to I_n using elementary row operations on the entire matrix $[A \; I_n]$. The result will be the matrix $[I_n \; A^{-1}]$. If this is not possible, then A is noninvertible (or singular).
3. Check your work by multiplying to see that $AA^{-1} = A^{-1}A = I_n$.

Example 2: Find the inverse of the matrix (if it exists), by solving the matrix equation $AX = I$.

$$A = \begin{bmatrix} 12 & 3 \\ 5 & -2 \end{bmatrix}$$

$AX = I$ if A is invertible $A^{-1}AX = A^{-1}I$

$X = A^{-1}$

$$[A \; I_2] = \left[\begin{array}{cc|cc} 12 & 3 & 1 & 0 \\ 5 & -2 & 0 & 1 \end{array}\right]$$

$-5R1 + 12R2 \rightarrow R2$

$$\left[\begin{array}{cc|cc} 12 & 3 & 1 & 0 \\ 0 & -39 & -5 & 12 \end{array}\right]$$

$13R1 + R2 \rightarrow R1$

$$\left[\begin{array}{cc|cc} 156 & 0 & 8 & 12 \\ 0 & -39 & -5 & 12 \end{array}\right]$$

$\frac{1}{348}R1 \rightarrow R1$

$$\left[\begin{array}{cc|cc} 1 & 0 & \frac{7}{174} & \frac{3}{29} \\ 0 & -39 & -5 & 12 \end{array}\right]$$

$-\frac{1}{29}R2 \rightarrow R2$

$$\left[\begin{array}{cc|cc} 1 & 0 & 2/39 & 1/13 \\ 0 & 1 & \frac{5}{39} & -\frac{4}{13} \end{array}\right] = [I_n \; A^{-1}]$$

$$\boxed{A^{-1} = \begin{bmatrix} 2/39 & 1/13 \\ 5/39 & -4/13 \end{bmatrix}}$$

Example 3: Find the inverse of the matrix (if it exists).

a. $A = \begin{bmatrix} 1 & -2 \\ 2 & -3 \end{bmatrix}$

b. $A = \begin{bmatrix} 10 & 5 & -7 \\ -5 & 1 & 4 \\ 3 & 2 & -2 \end{bmatrix}$

$$[A \mid I_3] = \left[\begin{array}{ccc|ccc} 10 & 5 & -7 & 1 & 0 & 0 \\ -5 & 1 & 4 & 0 & 1 & 0 \\ 3 & 2 & -2 & 0 & 0 & 1 \end{array}\right]$$

$R1 + 2R2 \to R2$

$$\left[\begin{array}{ccc|ccc} 10 & 5 & -7 & 1 & 0 & 0 \\ 0 & 7 & 1 & 1 & 2 & 0 \\ 3 & 2 & -2 & 0 & 0 & 1 \end{array}\right]$$

$-3R1 + 10R3 \to R3$

$$\left[\begin{array}{ccc|ccc} 10 & 5 & -7 & 1 & 0 & 0 \\ 0 & 7 & 1 & 1 & 2 & 0 \\ 0 & 5 & 1 & -3 & 0 & 10 \end{array}\right]$$

$-5R2 + 7R3 \to R3$

$$\left[\begin{array}{ccc|ccc} 10 & 5 & -7 & 1 & 0 & 0 \\ 0 & 7 & 1 & 1 & 2 & 0 \\ 0 & 0 & 2 & -26 & -10 & 70 \end{array}\right]$$

$$\tfrac{1}{2}R3 \to R3$$

$$\downarrow$$

$$\left[\begin{array}{ccc|ccc} 10 & 5 & -7 & 1 & 0 & 0 \\ 0 & 7 & 1 & 1 & 2 & 0 \\ 0 & 0 & 1 & -13 & -5 & 35 \end{array}\right]$$

$$-R3 + R2 \to R2$$

$$\downarrow$$

$$\left[\begin{array}{ccc|ccc} 10 & 5 & -7 & 1 & 0 & 0 \\ 0 & 7 & 0 & 14 & 7 & -35 \\ 0 & 0 & 1 & -13 & -5 & 35 \end{array}\right]$$

$$R1 + 7R3 \to R1$$

$$\downarrow$$

$$\left[\begin{array}{ccc|ccc} 10 & 5 & 0 & -90 & -35 & 245 \\ 0 & 7 & 0 & 14 & 7 & -35 \\ 0 & 0 & 1 & -13 & -5 & 35 \end{array}\right]$$

$$\tfrac{1}{7}R2 \to R2$$

$$\downarrow$$

$$\left[\begin{array}{ccc|ccc} 10 & 5 & 0 & -90 & -35 & 245 \\ 0 & 1 & 0 & 2 & 1 & -5 \\ 0 & 0 & 1 & -13 & -5 & 35 \end{array}\right]$$

$$-5R2 + R1 \to R1$$

$$\left[\begin{array}{ccc|ccc} 10 & 0 & 0 & -100 & -40 & 270 \\ 0 & 1 & 0 & 2 & 1 & -5 \\ 0 & 0 & 1 & -13 & -5 & 35 \end{array}\right]$$

$$\tfrac{1}{10}R1 \to R1$$

$$\downarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -10 & -4 & 27 \\ 0 & 1 & 0 & 2 & 1 & -5 \\ 0 & 0 & 1 & -13 & -5 & 35 \end{array}\right]$$

$$= \left[\, I_3 \mid A^{-1} \,\right]$$

$$\boxed{A^{-1} = \begin{bmatrix} -10 & -4 & 27 \\ 2 & 1 & -5 \\ -13 & -5 & 35 \end{bmatrix}}$$

THEOREM 2.8: PROPERTIES OF INVERSE MATRICES

If A is an invertible matrix, k is a positive integer, and c is a nonzero scalar, then A^{-1}, A^k, cA, and A^T are invertible and the following are true.

$BA^{-1} = IA^{-1}$

1. $\left(A^{-1}\right)^{-1}$ A

Proof:

Since A is invertible, we know $\exists B \ni AB = BA = I$. So $B = A^{-1}$ and $BA = A^{-1}A = I$. So A is the inverse of A^{-1}. //

2. $\left(A^{k}\right)^{-1} = \underbrace{A^{-1} \cdot A^{-1} \cdot A^{-1} \cdots A^{-1}}_{k \text{ times}} = (A^{-1})^k$

3. $\left(cA\right)^{-1} = \frac{1}{c}A^{-1}$

Proof:

$(cA)(\frac{1}{c}A^{-1}) = (c \cdot \frac{1}{c})(AA^{-1}) = 1 I_n = I_n$ ✓

$(\frac{1}{c}A^{-1})(cA) = (\frac{1}{c} \cdot c)(A^{-1}A) = 1 I_n = I_n$ ✓

4. $\left(A^{T}\right)^{-1} = (A^{-1})^T$

THEOREM 2.9: THE INVERSE OF A PRODUCT

If A and B are invertible matrices of order n, then AB is invertible and $\left(AB\right)^{-1} = B^{-1}A^{-1}$.

Proof:

$$\begin{aligned}(AB)(B^{-1}A^{-1}) &= A(BB^{-1})A^{-1} \\ &= AI_nA^{-1} \\ &= AA^{-1} \\ &= I_n \checkmark\end{aligned}$$

$$\begin{aligned}(B^{-1}A^{-1})(AB) &= B^{-1}(A^{-1}A)B \\ &= B^{-1}I_nB \\ &= B^{-1}B \\ &= I_n \checkmark\end{aligned}$$

Example 4: Use the inverse matrices below for the following problems.

$$A^{-1} = \begin{bmatrix} -\frac{2}{7} & \frac{1}{7} \\ \frac{3}{7} & \frac{2}{7} \end{bmatrix} \qquad B^{-1} = \begin{bmatrix} \frac{5}{11} & \frac{2}{11} \\ \frac{3}{11} & -\frac{1}{11} \end{bmatrix}$$

a. $(AB)^{-1}$ $= B^{-1}A^{-1}$

$$= \begin{bmatrix} 5/11 & 2/11 \\ 3/11 & -1/11 \end{bmatrix} \begin{bmatrix} -2/7 & 1/7 \\ 3/7 & 2/7 \end{bmatrix}$$

$$= \boxed{\begin{bmatrix} -4/77 & 9/77 \\ -9/77 & 1/77 \end{bmatrix}}$$

b. $(A^T)^{-1}$

$$= (A^{-1})^T$$

$$= \boxed{\begin{bmatrix} -2/7 & 3/7 \\ 1/7 & 2/7 \end{bmatrix}}$$

c. $(7A)^{-1}$ $= \frac{1}{7}A^{-1}$

$$= \frac{1}{7}\begin{bmatrix} -2/7 & 1/7 \\ 3/7 & 2/7 \end{bmatrix} \rightarrow = \boxed{\begin{bmatrix} -2/49 & 1/49 \\ 3/49 & 2/49 \end{bmatrix}}$$

THEOREM 2.10: CANCELLATION PROPERTIES

If C is an **invertible matrix**, then the following properties hold true.

1. If $AC = BC$ then $A = B$. Right cancellation property

Proof:

$ACC^{-1} = BCC^{-1}$ [C is invertible]

$AI = BI$

$A = B$ //

2. If $CA = CB$ then $A = B$. Left cancellation property

THEOREM 2.11: SYSTEMS OF EQUATIONS WITH UNIQUE SOLUTIONS

If A is an invertible matrix, then the system of linear equations $A\mathbf{x}=\mathbf{b}$ has a unique solution given by $\mathbf{x}=A^{-1}\mathbf{b}$.

Proof:

[A is invertible]

$A^{-1}A\vec{x} = A^{-1}\vec{b}$

$I\vec{x} = A^{-1}b$

$\vec{x} = A^{-1}\vec{b}$

A^{-1} is unique [Thm 2.7]. Suppose $\exists\ \vec{c} \ni \vec{x} = A^{-1}\vec{c}$. So,

$A\vec{x} = AA^{-1}\vec{c}$

$A\vec{x} = I\vec{c}$

$A\vec{x} = \vec{c}$.

Since $A\vec{x} = \vec{b}$, $\vec{c} = \vec{b}$. $\therefore$ $\vec{x} = A^{-1}\vec{b}$ is a unique solution to $A\vec{x} = \vec{b}$. //

CRYPTOGRAPHY

A cryptogram is a message written according to a secret code. Suppose we assign a number to each letter in the alphabet.

0	_	14	N
1	A	15	O
2	B	16	P
3	C	17	Q
4	D	18	R
5	E	19	S
6	F	20	T
7	G	21	U
8	H	22	V
9	I	23	W
10	J	24	X
11	K	25	Y
12	L	26	Z
13	M		

Example 5: Write the uncoded row matrices of size 1 x 3 for the message TARGET IS HOME.

$\vec{r}_1 = [20\ \ 1\ \ 18]$

$\vec{r}_2 = [7\ \ 5\ \ 20]$

$\vec{r}_3 = [0\ \ 9\ \ 19]$

$\vec{r}_4 = [0\ \ 8\ \ 15]$

$\vec{r}_5 = [13\ \ 5\ \ 0]$

Example 6: Use the following invertible matrix to encode the message TARGET IS HOME.

$$A = \begin{bmatrix} 1 & -2 & -2 \\ -1 & 1 & 3 \\ 1 & -1 & -4 \end{bmatrix}$$

3×3

$$\vec{r}_1 A = [20\ 1\ 18]\begin{bmatrix} 1 & -2 & -2 \\ -1 & 1 & 3 \\ 1 & -1 & -4 \end{bmatrix} = [37\ -57\ -109] = \vec{d}_1$$

$$\vec{r}_2 A = [22\ -29\ -79] = \vec{d}_2$$

$$\vec{r}_3 A = [10\ -10\ -49] = \vec{d}_3$$

$$\vec{r}_4 A = [7\ -7\ 36] = \vec{d}_4$$

$$\vec{r}_5 A = [8\ -21\ -11] = \vec{d}_5$$

Example 7: How would you decode a message?

$\vec{r}_i A = \vec{d}_i$ to encode

$r_i = \vec{d}_i A^{-1}$ to decode

$i = 1, 2, 3, 4, 5$

$$\boxed{\begin{matrix} 37 & -57 & -109 & 22 & -29 \\ -79 & 10 & -10 & -49 & 7 \\ -7 & 36 & 8 & -21 & -11 \end{matrix}}$$

DEFINITION OF AN ELEMENTARY MATRIX

An $n \times n$ matrix is called an elementary matrix when it can be obtained from the identity matrix I_n by a single elementary row operation.

Example 8: Identify the matrices that are elementary below.

$$A = \begin{bmatrix} 2 & -1 \\ 0 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 3 \\ 0 & 1 \\ -1 & -3 \end{bmatrix}$$

nope

$-2R2$ from I_3
$+ R3$ from I_3
yes !!

not square
so nope!

THEOREM 2.12: REPRESENTING ELEMENTARY ROW OPERATIONS

Let E be the elementary matrix obtained by performing an elementary row operation on I_m.

If that same elementary row operation is performed on an $m \times n$ matrix A, then the resulting matrix is given by the product EA.

Example 9: Given A and C below

$$A = \begin{bmatrix} 1 & 2 & -3 \\ 0 & 1 & 2 \\ -1 & 2 & 0 \end{bmatrix} \qquad C = \begin{bmatrix} 0 & 4 & -3 \\ 0 & 1 & 2 \\ -1 & 2 & 0 \end{bmatrix}$$

find an elementary matrix E such that $EA = C$.

$$EA = C$$

$$\begin{bmatrix} e_{11} & e_{12} & e_{13} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & -3 \\ 0 & 1 & 2 \\ -1 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 4 & -3 \\ 0 & 1 & 2 \\ -1 & 2 & 0 \end{bmatrix}$$

$e_{11} - e_{13} = 0 \rightarrow e_{11} = e_{13}$

$2e_{11} + e_{12} + 2e_{13} = 4$

$-3e_{11} + 2e_{12} = -3 \rightarrow e_{12} = \frac{1}{2}(3e_{11} - 3)$

$\rightarrow 2e_{11} + \frac{1}{2}(3e_{11} - 3) + 2e_{11} = 4$

$4e_{11} + \frac{3}{2}e_{11} - \frac{3}{2} = 4$

$\frac{11}{2}e_{11} = \frac{11}{2}$

$e_{11} = 1$

$e_{12} = \frac{1}{2}(3-3) = 0$

$e_{13} = 1$

$$\boxed{E = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Example 10: Find a sequence of elementary matrices that can be used to write the matrix in row-echelon form.

Equivalent matrix to A	Elementary Row Op,	Elementary Matrix
$A = \begin{bmatrix} 0 & 3 & -3 & 6 \\ 1 & -1 & 2 & -2 \\ 0 & 0 & 2 & 2 \end{bmatrix}$		
$\begin{bmatrix} 1 & -1 & 2 & -2 \\ 0 & 3 & -3 & 6 \\ 0 & 0 & 2 & 2 \end{bmatrix}$	$R_1 \leftrightarrow R_2$	$E_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
$\begin{bmatrix} 1 & -1 & 2 & -2 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 2 & 2 \end{bmatrix}$	$\frac{1}{3}R2$	$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
$\begin{bmatrix} 1 & -1 & 2 & -2 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix}$	$\frac{1}{2}R3$	$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{bmatrix}$

$= B$

This means that B is row-equivalent to A

$E_3 E_2 E_1 A = B$ yeo :) it checks out.

Furthermore,

$A = E_1^{-1} E_2^{-1} E_3^{-1} B$

DEFINITION OF ROW EQUIVALENCE

Let A and B be $m \times n$ matrices. Matrix B is row-equivalent to A when there exists a finite number of elementary matrices, $E_1, E_2, \ldots, E_k$ such that

$$B = E_k E_{k-1} E_{k-2} \cdots E_2 E_1 A$$

THEOREM 2.13: ELEMENTARY MATRICES ARE INVERTIBLE

If E is an elementary matrix, then E^{-1} exists and is an elementary matrix.

Example 11: Find the inverse of the elementary matrix.

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix}$$

Hmmm... to get E, on I_3 we computed $-3R2 + R3 \to R3$. So to undo it, we compute $3R2 + R3 \to R3$.

$$\boxed{E^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix}}$$

In general:

The sign changes on the entry from the row that didn't change and all entries in the changed row are multiplied by the reciprocal of the row that changed in E.

THEOREM 2.14: EQUIVALENT CONDITIONS

If A is an $n \times n$ matrix, then the following statements are equivalent.

1. A is invertible.
2. $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $n \times 1$ column matrix $\vec{b}$.
3. $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
4. A is row-equivalent to I_n.
5. A can be written as the product of elementary matrices.

Lower ↓ ↓ upper

THE *LU*-FACTORIZATION

3x3 lower Δ matrix

$$\begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

3x3 upper Δ matrix

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}$$

DEFINITION OF *LU*-FACTORIZATION

If the $n \times n$ matrix A can be written as the product of a lower triangular matrix L and an upper triangular matrix U, then $A = LU$ is an ***LU*-factorization** of A.

Example 12: Solve the linear system $A\mathbf{x} = \mathbf{b}$ by

1. Finding an *LU*-factorization of the coefficient matrix A.
2. Solving the lower triangular system $L\mathbf{y} = \mathbf{b}$.
3. Solving the upper triangular system $U\mathbf{x} = \mathbf{y}$.

$$I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{aligned} 2x_1 \quad\quad\quad &= 4 \\ -2x_1 + x_2 - x_3 &= -4 \\ 6x_1 + 2x_2 + x_3 &= 15 \\ -x_4 &= -1 \end{aligned}$$

2, 1, 1, 1

row ops

Elementary Matrices

1)

$$A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ -2 & 1 & -1 & 0 \\ 6 & 2 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

↓

$$\begin{bmatrix} 6 & 2 & 1 & 0 \\ -2 & 1 & -1 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$R1 \leftrightarrow R3$

$$E_1 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$E_1^{-1} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

↓

$$\begin{bmatrix} 6 & 2 & 1 & 0 \\ 0 & 5 & -2 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$1R1 + 3R2 \rightarrow R2$

$$E_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

↳ $-\frac{1}{3}R1 + \frac{1}{3}R2 \rightarrow R2$

from I_4

↳ $$E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1/3 & 1/3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

↓

$$\begin{bmatrix} 6 & 2 & 1 & 0 \\ 0 & 5 & -2 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$-3R3 + R1 \to R3$

$$E_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & -3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1/3 & 0 & -1/3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$\downarrow$

$$\begin{bmatrix} 6 & 2 & 1 & 0 \\ 0 & 5 & -2 & 0 \\ 0 & 0 & 9 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} = U$$

Flip signs

multiply all of R3 from I_4 by $\frac{1}{5}$

$-2R2 + \underline{\underline{5}}R3 \to R3$

inverse R3 : $\frac{1}{5}R3 + \frac{2}{5}R2$

$$E_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -2 & 5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$E_4^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2/5 & 1/5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$E_4 E_3 E_2 E_1 A = U$$

$$A = \underbrace{E_1^{-1} E_2^{-1} E_3^{-1} E_4^{-1}}_{L} U$$

$$A = LU$$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\frac{1}{3} & \frac{1}{3} & 0 & 0 \\ \frac{1}{3} & -\frac{2}{15} & -\frac{1}{15} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 6 & 2 & 1 & 0 \\ 0 & 5 & -2 & 0 \\ 0 & 0 & 9 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

check:

$$\begin{bmatrix} 6 & 2 & 1 & 0 \\ -2 & 1 & -1 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

see below for work

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$$\underbrace{E_1^{-1} E_2^{-1}} E_3^{-1} E_4^{-1}$$

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ -1/3 & 1/3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ -1/3 & 1/3 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{1}{3} & 0 & -\frac{1}{3} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1/3 & 0 & -\frac{1}{3} & 0 \\ -1/3 & 1/3 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2/5 & 1/5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{3} & -\frac{2}{15} & -\frac{1}{15} & 0 \\ -\frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Note: Since we $R1 \leftrightarrow R3$, for L, we need to $R1 \leftrightarrow R3$ for $\vec{b}$ to get $\vec{y}$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ -\frac{1}{3} & \frac{1}{3} & 0 & 0 \\ \frac{1}{3} & -\frac{2}{15} & -\frac{1}{15} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = L$$

---

2) $L\vec{y} = \vec{b}$

swapped R1 and R3 of $\vec{b}$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ -1/3 & 1/3 & 0 & 0 \\ 1/3 & -2/15 & -1/15 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 15 \\ -4 \\ 4 \\ -1 \end{bmatrix}$$

$y_1 = 15$

$y_2 = 3$

$y_3 = 9$

$y_4 = -1$

3) $U\vec{x} = \vec{y}$

$$\begin{bmatrix} 6 & 2 & 1 & 0 \\ 0 & 5 & -2 & 0 \\ 0 & 0 & 9 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 15 \\ 3 \\ 9 \\ 1 \end{bmatrix}$$

$x_1 = 2$

$x_2 = 1$

$x_3 = 1$

$x_4 = 1$

Suppose $E_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 4 & 0 & 0 & 2 \end{bmatrix}$, $I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$4R1 + 2R4 \to R4$

to get E_1^{-1}, a) opposite of coeff. of unchanging row,

b) mult by reciprocal of coeff in changing row

$$E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{-4}{2} & 0 & 0 & \frac{1}{2} \end{bmatrix}$$

$$E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -2 & 0 & 0 & \frac{1}{2} \end{bmatrix}$$

domain → V is $\mathbb{R}$

W is $\mathbb{R}$ ← codomain

$f(x) = 3x^2 - 1$ → range $-1 \le y$

input   output is y

$T: V \to W$

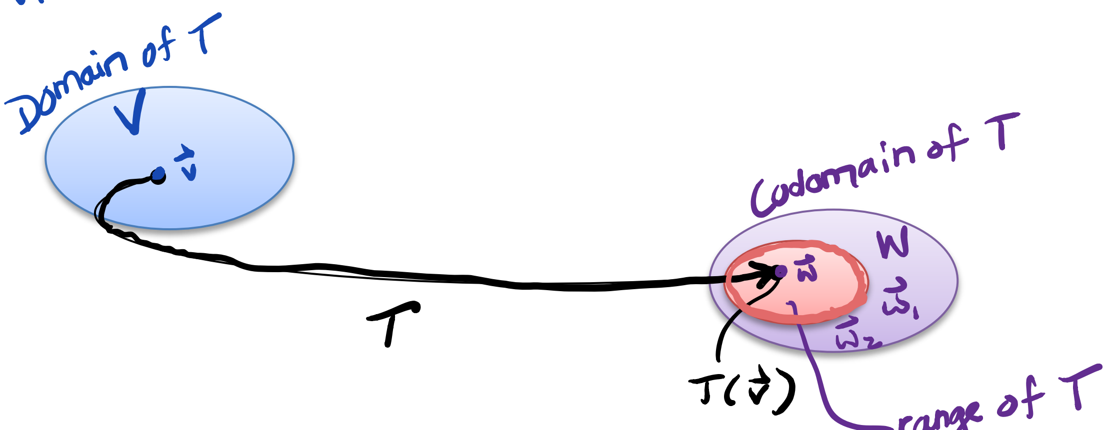
2.5: Linear Transformations

Learning Objectives

1. Find the preimage and image of a function
2. Determine if a function is a linear transformationWrite and use a stochastic matrix

IMAGES AND PREIMAGES OF FUNCTIONS

In this section we will learn about functions that __map__ a vector space __V__ onto a vector space __W__. This is denoted by __$T: V \to W$__. The standard function terminology is used for such functions. __V__ is called the __domain__ of __T__, and __W__ is called the __codomain__ of __T__. If $\mathbf{v}$ is in V, and $\mathbf{w}$ in W such that __$T(\vec{v}) = \vec{w}$__, __$\vec{v}$__ is called the __preimage__ of __$\vec{w}$__ under __T__. The set of all images of vectors in V is called the __range__ of __T__, and the set of all $\mathbf{v}$ in V such that __$T(\vec{v}) = \vec{w}$__ is called the __domain__ of __T__.



Example 1: Use the function to find (a) the image of $\mathbf{v}$ and (b) the preimage of $\mathbf{w}$.

$T(v_1, v_2) = (2v_2 - v_1, v_1, v_2)$, $\mathbf{v} = (0,6)$, $\mathbf{w} = (3,1,2)$

$T: \mathbb{R}^2 \to \mathbb{R}^3$

a) $T(0,6) = (2(6) - 0, 0, 6)$
$= (12, 0, 6)$

$(12,0,6)$ is the image of $\vec{v}$ under T.

b) $T(v_1, v_2) = (3,1,2)$ → $\vec{v} = (1,2)$ is the preimage of $\vec{w}$ under T.

$2v_2 - v_1 = 3$
$v_1 = 1$
$v_2 = 2$

DEFINITION OF A LINEAR TRANSFORMATION

Let V and W be vector spaces. The function $T: V \to W$ is called a linear transformation of V into W when the following two properties are true for all $\mathbf{u}$ and $\mathbf{v}$ in V and any scalar c.

1. $T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v})$
2. $T(c\vec{u}) = cT(\vec{u})$

A linear transformation is operation preserving because the same result occurs whether you perform the operations of addition and scalar multiplication before or after applying the linear transformation. Although the same symbols denote the vector operations in both V and W, you should note that the operations may be different.

Example 2: Determine whether the function is a linear transformation.

a. $T: R^3 \to R^3,\ T(x,y,z) = (x+1, y+1, z+1)$

$\vec{u} = (1,2,3),\ \vec{v} = (4,5,6)$

$T(\vec{u}+\vec{v}) \stackrel{?}{=} T(\vec{u}) + T(\vec{v})$

$T(5,7,9) \stackrel{?}{=} (2,3,4) + (5,6,7)$

$(6,8,10) \stackrel{?}{=} (7,9,11)$

No!

$\boxed{T \text{ is not a linear transformation}}$

b. $T: M_{2,2} \rightarrow R,\ T(A) = a + b + c + d$

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $B = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$, k is a scalar

$T(A+B) = T\left(\begin{bmatrix} a+e & b+f \\ c+g & d+h \end{bmatrix}\right) = (a+e) + (b+f) + (c+g) + (d+h)$

$= (a+b+c+d) + (e+f+g+h)$

$= T(A) + T(B)$ ✓

$T(kA) = T\begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix}$

$= ka + kb + kc + kd$

$= k(a+b+c+d)$

$= kT(A)$. ✓

yes, T is a linear transformation.

Exam 1 only goes through 2.4

THEOREM 2.15: PROPERTIES OF LINEAR TRANSFORMATIONS

Let T be a linear transformation from V into W, where $\mathbf{u}$ and $\mathbf{v}$ are in V. Then the following properties are true.

1. $T(\vec{0}) = \vec{0}$
2. $T(-\vec{v}) = -T(\vec{v})$
3. $T(\vec{u}-\vec{v}) = T(\vec{u}) - T(\vec{v})$

Proof:

$$T(\vec{u}-\vec{v}) = T(\vec{u}+(-\vec{v}))$$
$$= T(\vec{u}) + T(-\vec{v})$$
$$= T(\vec{u}) + T[-1(\vec{v})]$$
$$= T(\vec{u}) + (-T(\vec{v}))$$
$$= T(\vec{u}) - T(\vec{v}) \quad //$$

4. If $\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n$,

   then $T(\vec{v}) = T(c_1v_1 + c_2v_2 + \cdots + c_nv_n) = c_1T(\vec{v}_1) + c_2T(\vec{v}_2) + \cdots + c_nT(\vec{v}_n)$

Example 3: Let $T: R^3 \to R^3$ be a linear transformation such that $T(1,0,0) = (2,4,-1)$, $T(0,1,0) = (1,3,-2)$, and $T(0,0,1) = (0,-2,2)$. Find the indicated image.

$T(2,-1,0)$

$$(2,-1,0) = 2(1,0,0) - 1(0,1,0) + 0(0,0,1)$$

$$T[(2,-1,0)] = 2T[(1,0,0)] - 1T[(0,1,0)] + 0T[(0,0,1)]$$
$$= 2(2,4,-1) - (1,3,-2) + 0(0,-2,2)$$
$$= (4,8,-2) - (1,3,-2)$$
$$= \boxed{(3,5,0)}$$

THEOREM 2.16: THE LINEAR TRANSFORMATION GIVEN BY A MATRIX

Let A be an $m \times n$ matrix. The function T defined by

$$T(\vec{v}) = A\vec{v}$$

is a linear transformation from R^n into R^m. In order to conform to matrix multiplication with an $m \times n$ matrix, $n \times 1$ matrices represent the vectors in R^n and $m \times 1$ matrices represent the vectors in R^m.

$$A\mathbf{v} = \underbrace{\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}}_{m \times n} \overbrace{\begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}}^{n \times 1} = \begin{bmatrix} a_{11}v_1 + & \cdots & +a_{1n}v_n \\ \vdots & \ddots & \vdots \\ a_{m1}v_1 + & \cdots & +a_{mn}v_n \end{bmatrix}$$

Example 4: Define the linear transformation $T : R^n \to R^m$ by $T(\mathbf{v}) = A\mathbf{v}$. Find the dimensions of R^n and R^m.

a. $A = \begin{bmatrix} 1 & 2 \\ -2 & 4 \\ -2 & 2 \end{bmatrix}$

$T: R^n \to R^m \to T: R^2 \to R^3$

$$\boxed{\begin{aligned} R^n &= R^2 \\ R^m &= R^3 \end{aligned}}$$

b. $A = \begin{bmatrix} 1 & 3 & -1 & 0 \\ 0 & 1 & -2 & 1 \\ 2 & 1 & -4 & 1 \end{bmatrix}$

$T: R^n \to R^m$

$$\boxed{\begin{aligned} R^n &= R^4 \\ R^m &= R^3 \end{aligned}}$$

Example 5: Consider the linear transformation from Example 4, part a.

a. Find $T(2,4)$

$\vec{v} = (2,4)$

$T: R^2 \to R^3$

$$T(2,4) = A(2,4) = \begin{bmatrix} 1 & 2 \\ -2 & 4 \\ -2 & 2 \end{bmatrix}\begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

$$\boxed{T(2,4) = (10, 12, 4)}$$

$$= \begin{bmatrix} 10 \\ 12 \\ 4 \end{bmatrix}$$

$T(\vec{v}) = (v_1 + 2v_2, -2v_1 + 4v_2, -2v_1 + 2v_2)$

b. Find the preimage of $(-1,2,2)$

$T(\vec{v}) = A\vec{v} = \vec{w}$

$$\begin{bmatrix} 1 & 2 \\ -2 & 4 \\ -2 & 2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} \longrightarrow \left[\begin{array}{cc|c} 1 & 2 & -1 \\ -2 & 4 & 2 \\ -2 & 2 & 2 \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$\begin{cases} v_1 + 2v_2 = -1 \\ -2v_1 + 4v_2 = 2 \\ -2v_1 + 2v_2 = 2 \end{cases}$$

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

$$\boxed{T(-1,0) = (-1,2,2)}$$

c. Explain why the vector $(1,1,1)$ has no preimage under this transformation.

$$\left[\begin{array}{cc|c} 1 & 2 & 1 \\ -2 & 4 & 1 \\ -2 & 2 & 1 \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{array}\right]$$

$v_1 = 1$

$v_2 = 0$

$0 = -1$ False

$\vec{w} = (1,1,1) \in$ of the codomain, but not the range of T.

PART 2: DETERMINANTS, GENERAL VECTOR SPACES, AND MATRIX REPRESENTATIONS OF LINEAR TRANSFORMATIONS

3.1: THE DETERMINANT OF A MATRIX

Learning Objectives

1. Find the determinant of a 2 x 2 matrix
2. Find the minors and cofactors of a matrix
3. Use expansion by cofactors to find the determinant of a matrix
4. Find the determinant of a triangular matrix
5. Use elementary row operations to evaluate a determinant
6. Use elementary column operations to evaluate a determinant
7. Recognize conditions that yield zero determinants

Every square matrix can be associated with a real number called its determinant.

Historically, the use of determinants arose from the recognition of special patterns that occur in the solutions of systems of linear equations.

DEFINITION OF THE DETERMINANT OF A 2 x 2 MATRIX

The determinant of the matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

is given by $\det(A) = a_{11}a_{22} - a_{21}a_{12}$.

**Note: In this text, det(A) and |A| are used interchangeably to represent the determinant of a matrix. In this context, the vertical bars are used to represent the determinant of a matrix as opposed to the absolute value.

Example 1:

a. Find $\det(A)$ and $\det(B)$.

$$A = \begin{bmatrix} -1 & 4 \\ 11 & 7 \end{bmatrix} \qquad B = \begin{bmatrix} 21 & -3 \\ -6 & 10 \end{bmatrix}$$

$\det(A) = (-1)(7) - (11)(4)$

$= \boxed{-51}$

$\det(B) = (21)(10) - (-6)(-3)$

$= 210 - 18$

$= \boxed{192}$

the main diagonal entries are switched and the opposite sign is assigned to the other diagonal entries

Check this out…

$$A^{-1}=\frac{1}{\det(A)}\begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11}\end{bmatrix}$$

b. Find A^{-1} and B^{-1}

$A=\begin{bmatrix} -1 & 4 \\ 11 & 7\end{bmatrix}$

$A^{-1}=\frac{1}{-51}\begin{bmatrix} 7 & -4 \\ -11 & -1\end{bmatrix}$

$=\boxed{\begin{bmatrix} -7/51 & 4/51 \\ 11/51 & 1/51\end{bmatrix}}$

$B=\begin{bmatrix} 21 & -3 \\ -6 & 10\end{bmatrix}$

$B^{-1}=\frac{1}{192}\begin{bmatrix} 10 & 3 \\ 6 & 21\end{bmatrix}$

$=\boxed{\begin{bmatrix} 5/96 & 1/64 \\ 1/32 & 7/64\end{bmatrix}}$

DEFINITION OF MINORS AND COFACTORS OF A MATRIX

If A is a Square matrix, then the minor M_{ij} of the element a_{ij} is the determinant of the matrix obtained by deleting the ith row and the jth column of A. The cofactor C_{ij} is given by $C_{ij} = (-1)^{i+j} M_{ij}$.

Example 2: Find the minor and cofactor of a_{12} .and b_{13}.

a. $A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33}\end{bmatrix}$ →

$M_{12} = \det\begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33}\end{bmatrix} = a_{21}a_{33} - a_{31}a_{23}$

$C_{12} = (-1)^{1+2} M_{12} = \boxed{-(a_{21}a_{33} - a_{31}a_{23})}$

or $\boxed{(a_{31}a_{23} - a_{21}a_{33})}$

b. $B=\begin{bmatrix} 2 & 1 & 4 \\ 0 & 1 & 3 \\ 3 & -2 & 1\end{bmatrix}$

$C_{13} = (-1)^{1+3} M_{13}$

$C_{13} = 1 \det\begin{bmatrix} 0 & 1 \\ 3 & -2\end{bmatrix}$

$C_{13} = \boxed{-3}$

DEFINITION OF THE DETERMINANT OF A SQUARE MATRIX

If A is a square matrix of order $n > 2$, then the determinant of A is the sum of the entries in the first row of A multiplied by their respective cofactors. That is,

$$\det(A) = |A| = \sum_{j=1}^{n} a_{1j}C_{1j} = a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}.$$

Example 3: Confirm that, for 2x2 matrices, this definition yields $|A| = a_{11}a_{22} - a_{21}a_{12}$.

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$\begin{aligned} \det(A) &= a_{11}C_{11} + a_{12}C_{12} \\ &= a_{11}(-1)^{1+1}a_{22} + a_{12}(-1)^{1+2}a_{21} \\ &= a_{11}a_{22} - a_{21}a_{12} \checkmark \end{aligned}$$

Example 4: Find $|B|$.

$$B = \begin{bmatrix} 2 & -1 & 4 \\ 0 & 1 & 3 \\ 3 & -2 & 1 \end{bmatrix}$$

$$\begin{aligned} \det(B) &= a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} \\ &= 2(-1)^{1+1}\det\begin{bmatrix} 1 & 3 \\ -2 & 1 \end{bmatrix} + (-1)(-1)^{1+2}\det\begin{bmatrix} 0 & 3 \\ 3 & 1 \end{bmatrix} + 4(-1)^{1+3}\det\begin{bmatrix} 0 & 1 \\ 3 & -2 \end{bmatrix} \\ &= 2(7) - 1(-1)(-9) + 4(-3) \\ &= 14 - 9 - 12 \\ &= \boxed{-7} \end{aligned}$$

THEOREM 3.1: EXPANSION BY COFACTORS

If A be a square matrix of order n. Then the determinant of A is given by

$$\det(A) = |A| = \sum_{j=1}^{n} a_{ij}C_{ij} = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in} \quad (i\text{th row expansion})$$

$$\det(A) = |A| = \sum_{i=1}^{n} a_{ij}C_{ij} = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj} \quad (j\text{th column expansion})$$

Is there an easier way to complete the previous example?

$$B = \begin{bmatrix} 2 & -1 & 4 \\ 0 & 1 & 3 \\ 3 & -2 & 1 \end{bmatrix}$$

$$\det(B) = 0 \det\begin{bmatrix} -1 & 4 \\ -2 & 1 \end{bmatrix} + 1 \det\begin{bmatrix} 2 & 4 \\ 3 & 1 \end{bmatrix} - 3 \det\begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix}$$

$$= 0 + (-10) - 3(-1)$$

$$= \boxed{-7}$$

Alternative Method to evaluate the determinant of a 3 x 3 matrix: Copy the first and second columns of the matrix to form fourth and fifth columns. Then obtain the determinant by adding (or subtracting) the products of the six diagonals.

$$B = \begin{bmatrix} 2 & -1 & 4 \\ 0 & 1 & 3 \\ 3 & -2 & 1 \end{bmatrix}$$

$12 + -12 + 0 = 0$

$$\begin{matrix} 2 & -1 & 4 & 2 & -1 \\ 0 & 1 & 3 & 0 & 1 \\ 3 & -2 & 1 & 3 & -2 \end{matrix}$$

$2 - 9 + 0 = -7$

Bottom sum minus top sum

$-7 - 0 = -7$

$$\det(B) = \boxed{-7}$$

Example 5: Find $\det(A)$ and $\det(B)$.

$$A=\begin{array}{c} \begin{array}{cccc} + & - & + & - \end{array} \\ \begin{bmatrix} 1 & 0 & 2 & 6 \\ 3 & 7 & -1 & 0 \\ 6 & -1 & 2 & 5 \\ -3 & 5 & -8 & 7 \end{bmatrix} \end{array}$$

$-18+0+0+588=570$

$$\begin{matrix} 1 & 0 & 2 & 6 & 1 & 0 & 2 \\ 3 & 7 & -1 & 0 & 3 & 7 & -1 \\ 6 & -1 & 2 & 5 & 6 & -1 & 2 \\ -3 & 5 & -8 & 7 & 3 & 5 & -8 \end{matrix}$$

$98+0+0+144=242$

$242-570=-328$

$$\det(A)=1\det\begin{bmatrix} 7 & -1 & 0 \\ -1 & 2 & 5 \\ 5 & -8 & 7 \end{bmatrix}-0+2\det\begin{bmatrix} 3 & 7 & 0 \\ 6 & -1 & 5 \\ -3 & 5 & 7 \end{bmatrix}-6\det\begin{bmatrix} 3 & 7 & -1 \\ 6 & -1 & 2 \\ -3 & 5 & -8 \end{bmatrix}$$

$$=\boxed{-2210}$$

$$B=\begin{array}{c} \begin{array}{ccc} + & - & + \end{array} \\ \begin{bmatrix} 6 & 0 & 0 \\ 1 & 1 & 0 \\ 7 & -2 & 11 \end{bmatrix} \end{array}$$

$6(1)(11)=66$ ... it turns out that the determinant of a triangular matrix is the product of the elements on the main diagonal.

$$\det(B)=6\det\begin{bmatrix} 1 & 0 \\ -2 & 11 \end{bmatrix}-0+0$$

$$=6(11)$$

$$=\boxed{66}$$

What did you notice?

See above

THEOREM 3.2: DETERMINANT OF A TRIANGULAR MATRIX

If A is a triangular matrix of order n, then its determinant is the product of the elements on the main diagonal. That is, $\det(A) = |A| = a_{11}a_{22}a_{33}\cdots a_{nn}$.

Example 6: Find the values of λ, for which the determinant is zero.

$$\begin{vmatrix} \lambda-1 & 1 \\ 4 & \lambda-3 \end{vmatrix} = (\lambda-1)(\lambda-3)-4$$

$$0 = \lambda^2-4\lambda+3-4$$

$$0 = \lambda^2-4\lambda-1$$

$$\lambda = \frac{-(-4)\pm\sqrt{(-4)^2-4(1)(-1)}}{2(1)}$$

$$\lambda = \frac{4\pm\sqrt{20}}{2}$$

$$\lambda = \frac{4\pm2\sqrt{5}}{2}$$

$$\lambda = 2\pm\sqrt{5}$$

$$\boxed{\lambda_1 = 2-\sqrt{5},\ \lambda_2 = 2+\sqrt{5}}$$

Consider the following matrix:

$$A = \begin{bmatrix} -1 & 2 & 1 \\ 3 & 4 & -1 \\ 1 & 0 & 1 \end{bmatrix}$$

Find the determinant.

$$\det(A) = 1\det\begin{bmatrix} 2 & 1 \\ 4 & -1 \end{bmatrix} - 0 + 1\det\begin{bmatrix} -1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$= -6-10$$

$$= \boxed{-16}$$

Now let's put the matrix into row-echelon form. In other words, row reduce to an upper triangular matrix. Keep track of each elementary row operation.

$B = \begin{bmatrix} -1 & 2 & 1 \\ 3 & 4 & -1 \\ 1 & 0 & 1 \end{bmatrix}$

$|B| = |A| \rightarrow R_1 + R_3 \rightarrow R_3$

$\begin{bmatrix} -1 & 2 & 1 \\ 3 & 4 & -1 \\ 0 & 2 & 2 \end{bmatrix}$

$|B| = |A| \rightarrow 3R_1 + R_2 \rightarrow R_2$

$\begin{bmatrix} -1 & 2 & 1 \\ 0 & 10 & 2 \\ 0 & 2 & 2 \end{bmatrix}$

$-5R_3 + R_2 \rightarrow R_3 \leftarrow |B| = -5|A|$

$\begin{bmatrix} -1 & 2 & 1 \\ 0 & 10 & 2 \\ 0 & 0 & -8 \end{bmatrix}$

What's the determinant of this matrix?

$\det(B) = 80 \ldots$ $80 = -5(16)$

$\text{Det}(B) = -5\det(A)$

Take a closer look at the determinants of the two matrices. Do you notice anything?

THEOREM 3.3: ELEMENTARY ROW OPERATIONS AND DETERMINANTS

Let A and B be square matrices.

1. When B is obtained from A by interchanging (swapping) two rows of A, $|B| = -|A|$.

2. When B is obtained from A by add a multiple of a row of A to another row of A, $|B| = |A|$. To clarify, the "new" row is not scaled, but the row used to get the new row can be scaled. If the new row is scaled, you also use #3 below.

3. When B is obtained from A by multiplying a row of A by a nonzero constant c, $|B| = c|A|$.

NOTE: Theorem 3.3 remains valid when the word "column" replaces the word "row". Operations performed on columns are called elementary column operations.

Example 7: Determine which property of determinants the equation illustrates.

a. $\begin{vmatrix} 1 & -1 & 3 \\ 4 & 12 & 7 \\ 3 & -3 & 8 \end{vmatrix} = -\begin{vmatrix} 3 & -1 & 1 \\ 7 & 12 & 4 \\ 8 & -3 & 3 \end{vmatrix}$

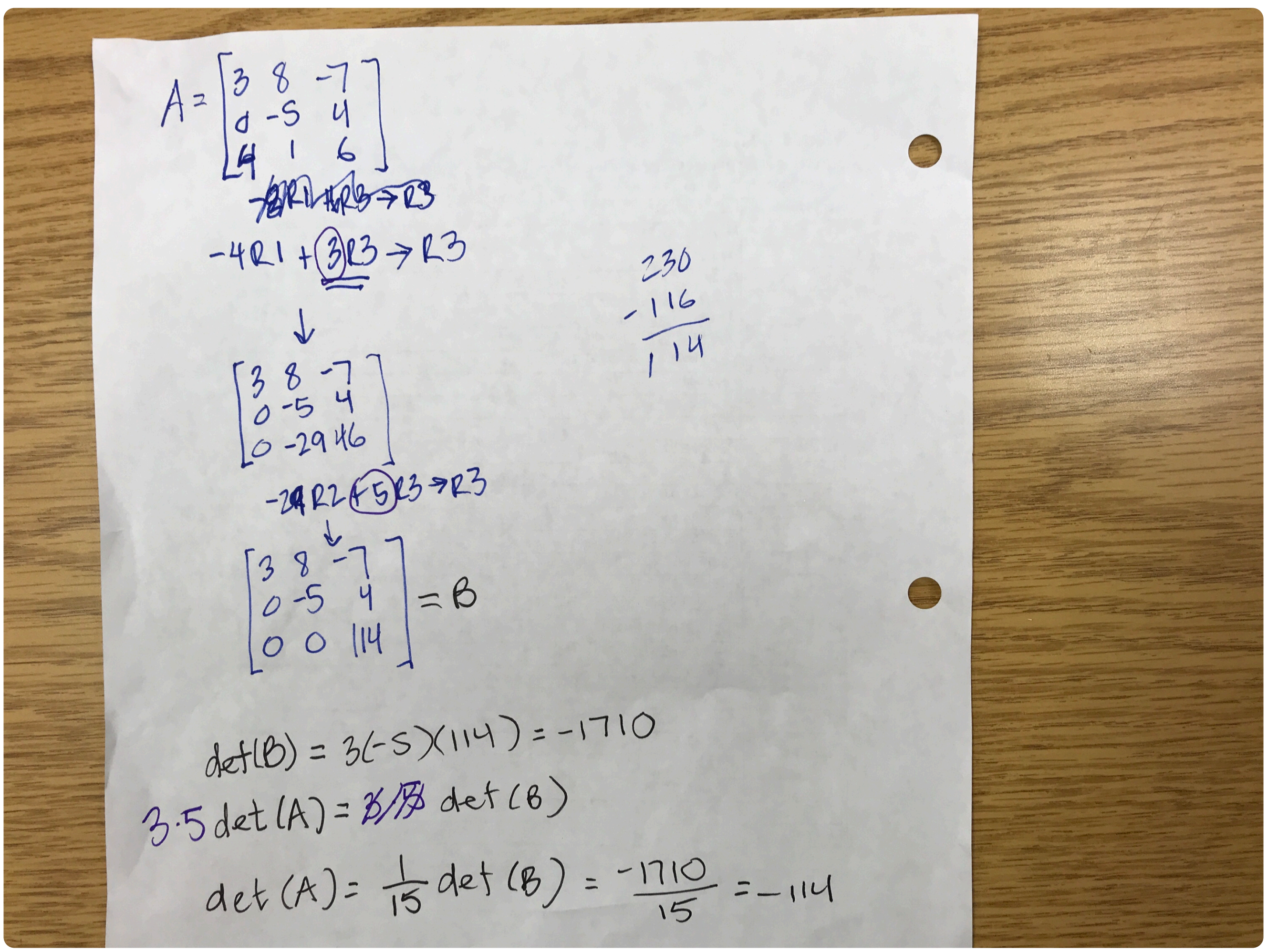
$C1 \leftrightarrow C3$

b. $\begin{vmatrix} 2 & -4 & 2 \\ 6 & 10 & 2 \\ 8 & -4 & 6 \end{vmatrix} = 8\begin{vmatrix} 1 & -2 & 1 \\ 3 & 5 & 1 \\ 4 & -2 & 3 \end{vmatrix}$

$2^3 = 8 \rightarrow$ a 2 from each row was brought outside the matrix.

Example 8: Use elementary row or column operations to find the determinant of the matrix.

$$A = \begin{bmatrix} 3 & 8 & -7 \\ 0 & -5 & 4 \\ 4 & 1 & 6 \end{bmatrix}$$



$A = \begin{bmatrix} 3 & 8 & -7 \\ 0 & -5 & 4 \\ 4 & 1 & 6 \end{bmatrix}$

~~$-2R1 + 4R3 \to R3$~~

$-4R1 + 3R3 \to R3$

$\downarrow$

$\begin{bmatrix} 3 & 8 & -7 \\ 0 & -5 & 4 \\ 0 & -29 & 46 \end{bmatrix}$

$-29R2 + 5R3 \to R3$

$\downarrow$

$\begin{bmatrix} 3 & 8 & -7 \\ 0 & -5 & 4 \\ 0 & 0 & 114 \end{bmatrix} = B$

$230 - 116 = 114$

$\det(B) = 3(-5)(114) = -1710$

$3 \cdot 5 \det(A) = \cancel{3/15} \det(B)$

$\det(A) = \frac{1}{15}\det(B) = \frac{-1710}{15} = -114$

THEOREM 3.4: CONDITIONS THAT YIELD A ZERO DETERMINANT

If A is a square matrix, and any one of the following conditions is true, then $\det(A) = 0$.

1. An entire row (or column) consists of zeros.
2. Two rows (or columns) are equal.
3. One row (or column) is a multiple of another row (or column).

	Cofactor Expansion		*Row Reduction*	
Order n	*Additions*	*Multiplications*	*Additions*	*Multiplications*
3	5	9	5	10
5	119	205	30	45
10	3,628,799	6,235,300	285	339

Example 9: Prove the property.

$$\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} = abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right),\ a \neq 0,\ b \neq 0,\ c \neq 0.$$

$$\begin{array}{ccc} + & - & + \end{array}$$

$$\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix}$$

$$= (1+a)\begin{vmatrix} 1+b & 1 \\ 1 & 1+c \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 1 & 1+c \end{vmatrix} + 1\begin{vmatrix} 1 & 1+b \\ 1 & 1 \end{vmatrix}$$

$$= (1+a)[(1+b)(1+c)-1] - [(1+c)-1] + [1-(1+b)]$$

$$= (1+a)(1+b)(1+c) - (1+a) - c - b$$

$$= (1+a+b+ab)(1+c) - 1 - a - c - b$$

$$= \cancel{1} + \cancel{c} + \cancel{a} + ac + \cancel{b} + bc + ab + abc - \cancel{1} - \cancel{a} - \cancel{b} - \cancel{c}$$

$$= abc\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + 1\right),\ a,b,c \neq 0 \quad //$$

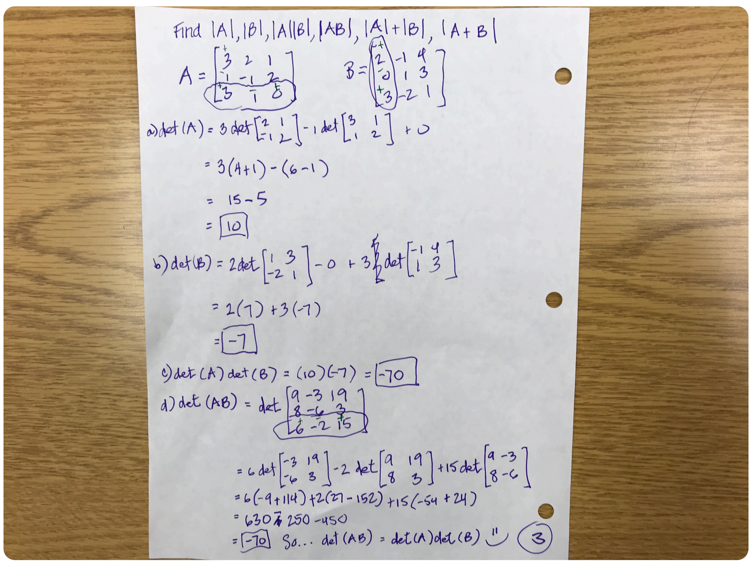
3.2: PROPERTIES OF DETERMINANTS

Learning Objectives

1. Find the determinant of a matrix product and a scalar multiple of a matrix
2. Find the determinant of an inverse matrix and recognize equivalent conditions for a nonsingular matrix
3. Find the determinant of the transpose of a matrix
4. Use Cramer's Rule to solve a system of linear equations
5. Use determinants to find area, volume, and equations of lines and planes

Example 1: Find $|A|, |B|, |A||B|, |A+B|, |A|+|B|$ and $|AB|$.

$$A = \begin{bmatrix} 3 & 2 & 1 \\ 1 & -1 & 2 \\ 3 & 1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & -1 & 4 \\ 0 & 1 & 3 \\ 3 & -2 & 1 \end{bmatrix}$$



Find $|A|, |B|, |A||B|, |AB|, |A|+|B|, |A+B|$

$A = \begin{bmatrix} 3 & 2 & 1 \\ 1 & -1 & 2 \\ 3 & 1 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 2 & -1 & 4 \\ 0 & 1 & 3 \\ 3 & -2 & 1 \end{bmatrix}$

a) $\det(A) = 3\det\begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix} - 1\det\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} + 0$

$= 3(4+1) - (6-1)$

$= 15 - 5$

$= \boxed{10}$

b) $\det(B) = 2\det\begin{bmatrix} 1 & 3 \\ -2 & 1 \end{bmatrix} - 0 + 3\det\begin{bmatrix} -1 & 4 \\ 1 & 3 \end{bmatrix}$

$= 2(7) + 3(-7)$

$= \boxed{-7}$

c) $\det(A)\det(B) = (10)(-7) = \boxed{-70}$

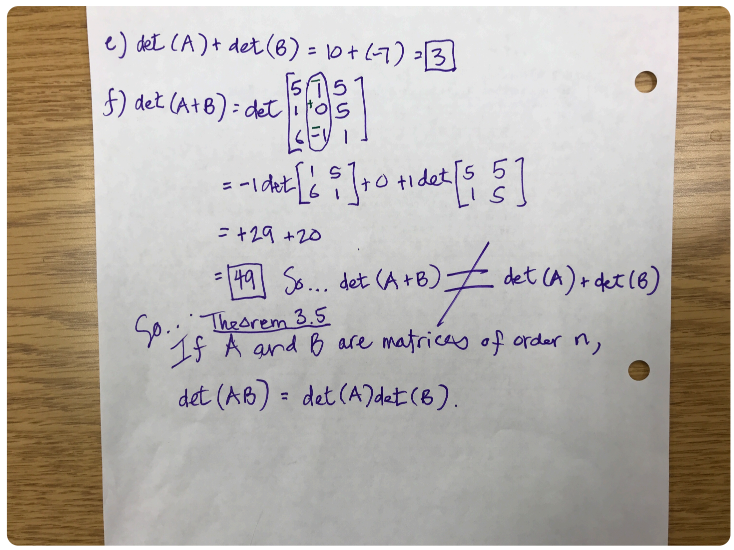
d) $\det(AB) = \det\begin{bmatrix} 9 & -3 & 19 \\ 8 & -6 & 3 \\ 6 & -2 & 15 \end{bmatrix}$

$= 6\det\begin{bmatrix} -3 & 19 \\ -6 & 3 \end{bmatrix} - 2\det\begin{bmatrix} 9 & 19 \\ 8 & 3 \end{bmatrix} + 15\det\begin{bmatrix} 9 & -3 \\ 8 & -6 \end{bmatrix}$

$= 6(-9+114) + 2(27-152) + 15(-54+24)$

$= 630 - 250 - 450$

$= \boxed{-70}$ So... $\det(AB) = \det(A)\det(B)$



e) $\det(A) + \det(B) = 10 + (-7) = \boxed{3}$

f) $\det(A+B) = \det\begin{bmatrix} 5 & 1 & 5 \\ 1 & 0 & 5 \\ 6 & -1 & 1 \end{bmatrix}$

$= -1\det\begin{bmatrix} 1 & 5 \\ 6 & 1 \end{bmatrix} + 0 + 1\det\begin{bmatrix} 5 & 5 \\ 1 & 5 \end{bmatrix}$

$= +29 + 20$

$= \boxed{49}$ So... $\det(A+B) \neq \det(A) + \det(B)$

So... Theorem 3.5

If A and B are matrices of order n,

$\det(AB) = \det(A)\det(B)$.

THEOREM 3.5: DETERMINANT OF A MATRIX PRODUCT

If A and B are square matrices of order n, then

$$\det(AB) = \det(A)\det(B)$$

Example 2: Find $|3A|$ and $|3B|$.

$A = \begin{bmatrix} 1 & -1 \\ 3 & 10 \end{bmatrix}$ 2×2

$B = \begin{bmatrix} 2 & -1 & 4 \\ 0 & 1 & 3 \\ 3 & -2 & 1 \end{bmatrix}$ 3×3

$|A| = 13$

$|3A| = \begin{vmatrix} 3 & -3 \\ 9 & 30 \end{vmatrix}$

$= 117$

$= 3\cdot 3\cdot 13$

maybe $|3A|$

yep! $= 3^2|A|$

$|B| = -7$

Is $|3B| = 3^3\cdot(-7)$

$|3B| = \begin{vmatrix} 6 & -3 & 12 \\ 0 & 3 & 9 \\ 9 & -6 & 3 \end{vmatrix}$

$= 0 + 3\begin{vmatrix} 6 & 12 \\ 9 & 3 \end{vmatrix} - 9\begin{vmatrix} 6 & -3 \\ 9 & -6 \end{vmatrix}$

$= 3(18-108) - 9(-36+27)$

$= -270 + 81$

$= -189$

$= -3\cdot 3\cdot 3\cdot 7$

$= 3^3(-7)$

THEOREM 3.6: DETERMINANT OF A SCALAR MULTIPLE OF A MATRIX

If A is a square matrix of order n and c is a scalar, then the determinant of $|cA|$ is

$c^n \det(A)$

Proof:

Let $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$, $cA = \begin{bmatrix} ca_{11} & ca_{12} & \cdots & ca_{1n} \\ ca_{21} & ca_{22} & \cdots & ca_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ ca_{n1} & ca_{n2} & \cdots & ca_{nn} \end{bmatrix}$, $a_{ij} \in \mathbb{R}$, $1 \le i \le n$ and $1 \le j \le n$.

$\det(A) = \sum_{j=1}^{n} a_{ij} C_{ij}$

$\det(cA) = \sum_{j=1}^{n} ca_{ij} c^{n-1} C_{ij} = ca_{11}c^{n-1}C_{11} + ca_{12}c^{n-1}C_{12} + \cdots + ca_{1n}c^{n-1}C_{12}$

$= \sum_{j=1}^{n} c^n a_{ij} C_{ij}$

$= c^n \det(A)$

$= c^n \sum a_{ij} C_{ij}$

Example 3: Find A^{-1}, $|A|$, $|A^{-1}|$, B^{-1}, $|B^{-1}|$, and $|B|$.

$A = \begin{bmatrix} -3 & 6 \\ -2 & 4 \end{bmatrix}$ $B = \begin{bmatrix} 5 & 2 \\ 11 & 7 \end{bmatrix}$

$|A| = -12 + 12 = 0$

So A is singular, A^{-1} DNE

$|B| = 35 - 22 = \boxed{13}$

$B^{-1} = \frac{1}{|B|}\begin{bmatrix} 7 & -2 \\ -11 & 5 \end{bmatrix} = \begin{bmatrix} 7/13 & -2/13 \\ -11/13 & 5/13 \end{bmatrix}$

$|B^{-1}| = \frac{35}{169} - \frac{22}{169} = \frac{13}{169} = \frac{1}{13}$

$\downarrow$

$\frac{1}{\det(B)}$

THEOREM 3.7: DETERMINANT OF AN INVERTIBLE MATRIX

A square matrix A is invertible (nonsingular) if and only if

$\det(A) \neq 0$

Example 4: Find $|A|$ and $|A^{-1}|$.

$A = \begin{bmatrix} -3 & 3 \\ -2 & 1 \end{bmatrix}$

$\det(A) = -3 + 6 = 3$

$\det(A^{-1}) = \frac{1}{\det(A)} = \boxed{\frac{1}{3}}$

THEOREM 3.8: DETERMINANT OF AN INVERSE MATRIX

If A is an $n \times n$ invertible matrix, then

$$\det(A^{-1}) = \frac{1}{\det(A)}$$

Proof:

Since A is invertible, $\exists\, A^{-1} \ni AA^{-1} = I_n = A^{-1}A$, and $\det(A) \neq 0$. [Thm 3.7] $\det(AA^{-1}) = \det(A)\det(A^{-1})$, and $\det(AA^{-1}) = \det(I_n) = 1$. So $\det(A)\det(A^{-1}) = 1$, [Thm 3.5] and $\det(A^{-1}) = \frac{1}{\det(A)}$. //

EQUIVALENT CONDITIONS FOR A NONSINGULAR MATRIX

If A is an $n \times n$ matrix, then the following statements are equivalent.

1. A is invertible.
2. $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $n \times 1$ column matrix.
3. $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
4. A is row-equivalent to I_n.
5. A can be written as the product of elementary matrices.
6. $\det(A) \neq 0$.

Example 5: Determine if the system of linear equations has a unique solution.

$$\begin{aligned} x_1 + x_2 - x_3 &= 4 \\ 2x_1 - x_2 - x_3 &= 6 \\ 3x_1 - 2x_2 + 2x_3 &= 0 \end{aligned}$$

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & -1 & -1 \\ 3 & -2 & 2 \end{bmatrix}$$

$\det(A) = -10 \neq 0$, $\therefore \exists$ a unique solution to this system.

Example 6: Find $|A|$ and $|A^T|$.

$$A=\begin{bmatrix} 7 & 12 \\ 2 & -2 \end{bmatrix}$$

$\det(A) = -14 - 24 = -38$

$A^T = \begin{bmatrix} 7 & 2 \\ 12 & -2 \end{bmatrix}$ $\det(A^T) = -14 - 24 = -38$

THEOREM 3.9: DETERMINANT OF A TRANSPOSE

If A is a square matrix, then

$$\det(A^T) = \det(A)$$

Example 7: Solve the system of linear equations. Assume that $a_{11}a_{22} - a_{21}a_{12} \neq 0$.

$$a_{11}x_1 + a_{12}x_2 = b_1 \quad (A)$$
$$a_{21}x_1 + a_{22}x_2 = b_2 \quad (B)$$

$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$

1) Isolate x_1 from A and then sub. into B.

a) $a_{11}x_1 + a_{12}x_2 = b_1$

$a_{11}x_1 = b_1 - a_{12}x_2$

$x_1 = \dfrac{b_1 - a_{12}x_2}{a_{11}}$

b) $a_{21}\left(\dfrac{b_1 - a_{12}x_2}{a_{11}}\right) + \dfrac{a_{11}}{a_{11}}a_{22}x_2 = b_2$

$\dfrac{a_{21}b_1 - a_{21}a_{12}x_2}{a_{11}} + \dfrac{a_{11}a_{22}x_2}{a_{11}} = b_2$

$\dfrac{a_{21}b_1 - a_{21}a_{12}x_2 + a_{11}a_{22}x_2}{a_{11}} = b_2$

$a_{21}b_1 - a_{21}a_{12}x_2 + a_{11}a_{22}x_2 = a_{11}b_2$

$x_2(a_{11}a_{22} - a_{21}a_{12}) = a_{11}b_2 - a_{21}b_1$

$x_2 = \dfrac{a_{11}b_2 - a_{21}b_1}{a_{11}a_{22} - a_{21}a_{12}}$

2) Sub x_2 into eq. A

$\dfrac{(a_{11}a_{22} - a_{21}a_{12})(a_{11}x_1)}{a_{11}a_{22} - a_{21}a_{12}} + a_{12}\left[\dfrac{a_{11}b_2 - a_{21}b_1}{a_{11}a_{22} - a_{21}a_{12}}\right] = b_1$

$$\frac{a_{11}^2 a_{22} x_1 - a_{21} a_{12} a_{11} x_1 + a_{12} a_{11} b_2 - a_{21} a_{12} b_1}{a_{11} a_{22} - a_{21} a_{12}} = b_1$$

$$x_1 (a_{11}^2 a_{22} - a_{21} a_{12} a_{11}) + a_{12} a_{11} b_2 - a_{21} a_{12} b_1 = b_1 (a_{11} a_{22} - a_{21} a_{12})$$

$$x_1 = \frac{\cancel{a_{21} a_{12}} b_1 + b_1 a_{11} a_{22} - b_1 \cancel{a_{21}} a_{12} - a_{11} a_{12} b_2}{a_{11} (a_{11} a_{22} - a_{21} a_{12})}$$

$$x_1 = \frac{\cancel{a_{11}} (b_1 a_{22} - a_{12} b_2)}{\cancel{a_{11}} (a_{11} a_{22} - a_{21} a_{12}}$$

$$x_1 = \frac{b_1 a_{22} - a_{12} b_2}{a_{11} a_{22} - a_{21} a_{12}} \qquad x_2 = \frac{a_{11} b_2 - a_{21} b_1}{a_{11} a_{22} - a_{21} a_{12}}$$

$$x_1 = \frac{\det(A_1)}{\det(A)} \qquad x_2 = \frac{\det(A_2)}{\det(A)}$$

$$A_1 = \begin{bmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{bmatrix}$$

$$A_2 = \begin{bmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{bmatrix}$$

THEOREM 3.10: CRAMER'S RULE

If a system of n linear equations in n variables has a coefficient matrix A with a nonzero determinant $|A|$, then the solution of the system is

$$x_1 = \frac{\det(A_1)}{\det(A)},\ x_2 = \frac{\det(A_2)}{\det(A)},\ \ldots,\ x_n = \frac{\det(A_n)}{\det(A)}$$

Where the jth column of A_j is the column of constants in the system of equations.

Example 8: If possible, use Cramer's Rule to solve the system.

a.

$$-x_1 - 2x_2 = 7$$
$$2x_1 + 4x_2 = 11$$

Cramer's Rule does not apply

$$A = \begin{bmatrix} -1 & -2 \\ 2 & 4 \end{bmatrix}$$

$$\det(A) = -4+4 = 0$$

b.

$$-8x_1 + 7x_2 - 10x_3 = -151$$
$$12x_1 + 3x_2 - 5x_3 = 86$$
$$15x_1 - 9x_2 + 2x_3 = 187$$

$$A = \begin{bmatrix} -8 & 7 & -10 \\ 12 & 3 & -5 \\ 15 & -9 & 2 \end{bmatrix}$$

$$\vec{b} = \begin{bmatrix} -151 \\ 86 \\ 187 \end{bmatrix}$$

$$\det(A) = 1149$$

$$A_1 = \begin{bmatrix} -151 & 7 & -10 \\ 86 & 3 & -5 \\ 187 & -9 & 2 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} -8 & -151 & -10 \\ 12 & 86 & -5 \\ 15 & 187 & 2 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} -8 & 7 & -151 \\ 12 & 3 & 86 \\ 15 & -9 & 187 \end{bmatrix}$$

$$x_1 = \frac{\det(A_1)}{\det(A)} = \frac{11490}{1149} = 10$$

$$x_2 = \frac{\det(A_2)}{\det(A)} = \frac{-3447}{1149} = -3$$

$$x_3 = \frac{\det(A_3)}{\det(A)} = \frac{5745}{1149} = 5$$

$\boxed{\{(10,-3,5)\}\ \text{consistent}}$

AREA OF A TRIANGLE IN THE *xy*-PLANE

The area of a triangle with vertices (x_1, y_1), (x_2, y_2), and (x_3, y_3) is +

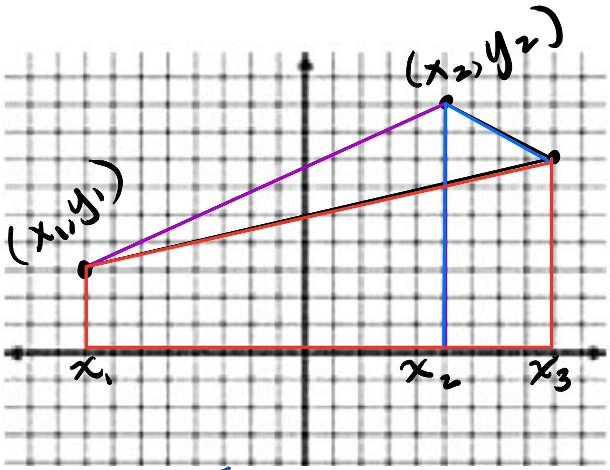
$$\text{Area} = \pm\frac{1}{2}\det\begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix}$$

$$1(x_2y_3 - y_2x_3) - 1(x_1y_3 - y_1x_3) + 1(x_1y_2 - y_1x_2)$$

where the sign ($\pm$) is chosen to give positive area.

$$\text{Area}_{Trap} = \frac{b_1+b_2}{2}h$$

Proof:



$$\text{Area}_{Trap1} = \frac{y_1+y_2}{2}(x_2-x_1)$$

$$\text{Area}_{Trap2} = \frac{y_2+y_3}{2}(x_3-x_2)$$

$$\text{Area}_{Trap3} = \frac{y_1+y_3}{2}(x_3-x_1)$$

$$A_\Delta = \frac{1}{2}\left[(x_2-x_1)(y_1+y_2) + (x_3-x_2)(y_2+y_3) - (x_3-x_1)(y_1+y_3)\right]$$

Example 9: Find the area of the triangle whose vertices are $(1,-1)$, $(3,-5)$, and $(0,-2)$.

TEST FOR COLLINEAR POINTS IN THE *xy*-PLANE

Three points (x_1, y_1), (x_2, y_2), and (x_3, y_3) are collinear if and only if

$$\det\begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix} = 0$$

TWO-POINT FORM OF THE EQUATION OF A LINE

An equation of the line passing through the distinct points (x_1, y_1) and (x_2, y_2) is given by

$$\det\begin{bmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{bmatrix} = 0$$

VOLUME OF A TETRAHEDRON

The volume of a tetrahedron with vertices (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3), and (x_4, y_4, z_4) is

$$V = \pm \frac{1}{6}\det\begin{bmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{bmatrix}$$

where the sign ($\pm$) is chosen to give positive volume.

Example 11: Find the volume of the tetrahedron with vertices $(1,1,1)$, $(0,0,0)$, $(2,1,-1)$, and $(-1,1,2)$.

$$V = \pm\frac{1}{6}\begin{vmatrix} 1 & 1 & 1 & 1 \\ \cancel{0} & \cancel{0} & \cancel{0} & 1 \\ 2 & 1 & -1 & 1 \\ -1 & 1 & 2 & 1 \end{vmatrix}$$

$$= \pm\frac{1}{6}\det\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ -1 & 1 & 2 \end{bmatrix}$$

$$= \pm\frac{1}{6}\left[-(\overset{3}{4-1}) + (\overset{3}{2+1}) - (\overset{-3}{-1-2})\right]$$

$$= \pm\frac{1}{6}(3) = \boxed{\frac{1}{2}\text{ cubic unit}}$$

TEST FOR COPLANAR POINTS IN SPACE

Four points, (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3), and (x_4, y_4, z_4) are coplanar if and only if

$$\det\begin{bmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{bmatrix} = 0$$

THREE-POINT FORM OF THE EQUATION OF A LINE

An equation of the plane passing through the distinct points (x_1, y_1, z_1), (x_2, y_2, z_2), and (x_3, y_3, z_3) is given by

$$\det\begin{bmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{bmatrix} = 0$$

3.3: GENERAL VECTOR SPACES

Learning Objectives:

1. Determine whether a set of vectors is a vector space
2. Determine if a subset of a known vector space V is a subspace of V
3. Write a vector as a linear combination of other vectors
4. Recognize bases in the vector spaces R^n, P_n, and $M_{m,n}$
5. Determine whether a set S of vectors in a vector space V is a basis for V
6. Find the dimension of a vector space

DEFINITION OF A VECTOR SPACE

Let V be a set on which two operations (vector addition and scalar multiplication) are defined. If the listed axioms are satisfied for every $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in V and every scalar (real number) c and d, then V is called a **vector space**.

Addition

1. $\mathbf{u}+\mathbf{v}$ is in V. — closure under addition
2. $\mathbf{u}+\mathbf{v} = \vec{v}+\vec{u}$ — commutative property
3. $\mathbf{u}+(\mathbf{v}+\mathbf{w}) = (\vec{u}+\vec{v})+\vec{w}$ — associative property
4. V has a zero vector $\vec{0}$ such that for every $\vec{v}$ in V, $\vec{v}+\vec{0}=\vec{v}$. — additive identity
5. For every $\vec{v}$ in V, there is a vector in V denoted by $-\vec{v}$ such that $\vec{v}+(-\vec{v})=\vec{0}$. — additive inverse

Scalar Multiplication

6. $c\mathbf{u}$ is in V. — closure under scalar mult.
7. $c(\mathbf{u}+\mathbf{v}) = c\vec{u}+c\vec{v}$ — distributive property
8. $(c+d)\mathbf{u} = c\vec{u}+d\vec{u}$ — distributive property
9. $c(d\mathbf{u}) = (cd)\vec{u}$ — associative property
10. $1(\mathbf{u}) = \vec{u}$ — scal. multiplicative identity

THEOREM 3.11: PROPERTIES OF SCALAR MULTIPLICATION

Let $\mathbf{v}$ be any element of a vector space V, and let c be any scalar. Then the following properties are true.

1. $0\mathbf{v} =$ $\vec{0}$
2. $c\mathbf{0} =$ $\vec{0}$
3. If $c\vec{v}=\vec{0}$, then $c=0$ or $\vec{v}=\vec{0}$.
4. $(-1)\mathbf{v} =$ $-\vec{v}$

Example 1: Determine whether the set, together with the indicated operations, is a vector space. If it is not, then identify at least one of the ten vector space axioms that fails.

a. The set of all 2 x 2 matrices of the form $S = \left\{ \begin{bmatrix} a & b \\ c & 1 \end{bmatrix} : a,b,c,d \in R \right\}$.

$A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 4 & 5 \\ 6 & 1 \end{bmatrix} \in S$ and $A+B = \begin{bmatrix} 5 & 7 \\ 9 & 2 \end{bmatrix} \notin S$.

S is not closed under addition.

b. The set of all 2 x 2 nonsingular matrices with the standard operations.

Axiom 1 $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $-A = \begin{bmatrix} -1 & -2 \\ -3 & -4 \end{bmatrix}$ are nonsingular [nonzero determinants]

$A+(-A) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ which is singular so set isn't closed under +.

Axiom 4 Also, $\vec{0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is not in this set, so $\nexists\, \vec{0} \ni A+\vec{0} = A$.

IMPORTANT VECTOR SPACES CONTINUED

$C(-\infty,\infty)$ = the set of all continuous functions defined on the real number line.

$C[a,b]$ = the set of all continuous functions defined on a closed interval $[a,b]$.

P = the set of all polynomials.

P_n = the set of all polynomials of degree $\leq n$.

$M_{m,n}$ = the set of all $m \times n$ matrices.

$M_{n,n}$ = the set of all $n \times n$ square matrices.

Example 2: Describe the zero vector (the additive identity) of the vector space.

a. $C(-\infty,\infty)$

$\vec{0}: f(x)=0$

$[y=0]$

b. $M_{1,4}$

$\vec{0}=[0\ 0\ 0\ 0]$

Example 3: Describe the additive inverse of a vector in the vector space.

a. $C(-\infty,\infty)$

$\boxed{-f(x)}$

b. $M_{1,4}$

If $A=[a_{11}\ a_{12}\ a_{13}\ a_{14}]$

$\boxed{-A=[-a_{11}\ -a_{12}\ -a_{13}\ -a_{14}]}$

Example 4: Determine whether the set of continuous functions, $C(-\infty,\infty)$ is a vector space.

Let $f, g, h \in C(-\infty,\infty)$ and $c, d \in R$.

1. Closure under addition.

$f(x)+g(x)=(f+g)(x) \in C(-\infty,\infty)$ ✓

2. Commutativity under addition.

$$\begin{aligned}(f+g)(x) &= f(x)+g(x)\\ &= g(x)+f(x)\\ &= (g+f)(x)\ ✓\end{aligned}$$

3. Associativity under addition.

$$\begin{aligned}f(x)+(g+h)(x) &= f(x)+[g(x)+h(x)]\\ &=[f(x)+g(x)]+h(x)\\ &=(f+g)(x)+h(x)\ ✓\end{aligned}$$

4. Additive identity.

$$f(x)+\vec{0} = f(x)+0$$
$$= f(x) \checkmark$$

$$c\vec{u} = c(u_1, u_2)$$
$$= (cu_1, cu_2)$$

5. Additive inverse.

$$[f+(-f)](x) = f(x)+[-f(x)]$$
$$= 0$$
$$= \vec{0} \checkmark$$

6. Closure under scalar multiplication.

$$cf(x) = \underbrace{(cf)}(x) \in C(-\infty,\infty) \checkmark$$

7. Distributivity under scalar multiplication (2 vectors and 1 scalar).

$$[c(f+g)(x)] = c[(f+g)(x)]$$
$$= c[f(x)+g(x)]$$
$$= cf(x)+cg(x) \checkmark$$

8. Distributivity under scalar multiplication (2 scalars and 1 vector).

$$[(c+d)f](x) = (c+d)f(x)$$
$$= cf(x)+df(x) \checkmark$$

9. Associativity under scalar multiplication.

$$[c(df)](x) = c[df](x)$$
$$= c(df(x))$$
$$= (cd)f(x) \checkmark$$

10. Scalar multiplicative identity.

$$(1f)(x) = 1f(x)$$
$$= f(x) \checkmark$$

Conclusion? $C(-\infty,\infty)$ is a vector space.

Example 5: Determine whether the set W is a subspace of the vector space V with the standard operations of addition and scalar multiplication.

a. $V : C[-1,1]$

W : The set of all functions that are differentiable on $[-1,1]$

W is a nonempty subset of V [diff. → continuity].

Let f and g ∈ W, and let c ∈ ℝ.

$$\frac{d}{dx}[f(x)] + \frac{d}{dx}[g(x)] = \frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}[(f+g)(x)] \checkmark$$

$$c\frac{d}{dx}f(x) = \frac{d}{dx}[cf(x)] \checkmark$$

∴ W is a subspace of V.

b. $V : C(-\infty, \infty)$

W : The set of all negative functions: $f(x) < 0$.

$f(x) = -x^2 < 0$

$c = -5$

$cf(x) = -5(-x^2) = 5x^2 > 0$

W is not closed under scalar mult.

c. $V : C(-\infty, \infty)$

W : The set of all odd cont functions: $f(-x) = -f(x)$.

checking

$f(x) = x$

$g(x) = \sin x$

$(f+g)(-x) \stackrel{?}{=} -(f+g)(x)$

$-x + \sin(-x) \stackrel{?}{=} -(x + \sin x)$

$-(x + \sin x) = -(x + \sin x)$ ✓

W is a nonempty subset of V.

Let f, g be odd functions, and let $c \in \mathbb{R}$

$(f+g)(-x) = f(-x) + g(-x)$

$= -f(x) + (-g(x))$

$(cf)(-x) = cf(-x)$

$= c[-f(x)]$

$= -cf(x)$ ✓

$= -(f(x) + g(x))$

$= -(f+g)(x)$ ✓

$\therefore$ W is subspace of V.

d. $V:\{M_{n,n}:n\in Z^+\}$

W : The set of all n x n diagonal matrices.

W is a nonempty subset of V. Let $A = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & a_{nn} \end{bmatrix}$,

$B = \begin{bmatrix} b_{11} & 0 & \cdots & 0 \\ 0 & b_{22} & 0 \cdots & 0 \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & b_{nn} \end{bmatrix}$, $c \in R$.

$A+B = \begin{bmatrix} a_{11}+b_{11} & 0 & \cdots & 0 \\ 0 & a_{22}+b_{22} & 0 \cdots & 0 \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & a_{nn}+b_{nn} \end{bmatrix} \in W$ ✓

$cA = c\begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & 0 \cdots & 0 \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & a_{nn} \end{bmatrix}$

$= \begin{bmatrix} ca_{11} & 0 & 0 \cdots & 0 \\ 0 & ca_{22} & 0 \cdots & 0 \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & ca_{nn} \end{bmatrix} \in W$ ✓

$\therefore$ W is a subspace of V

e. W : The set of all n x n matrices whose trace is nonzero.

$V:\{M_{n,n}:n\in Z^+\}$

$0\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \leftarrow$ trace $= 0$

$\uparrow$

trace $= 1+5+9 = 15 \neq 0$

$\boxed{\text{not closed under scal. mult.}}$

f. $V:C(-\infty,\infty)$

$W:\{ax+b:a,b\in R, a\neq 0\}$

$f(x) = 2x+5$

$g(x) = -2x-1$

$(f+g)(x) = (2x+5)+(-2x-1)$

$= 0x+4 \notin W$.

$\boxed{\text{not closed under addition}}$

g. $V:\{M_{m,n}: m,n \in Z^+\}$

$W:\left\{\begin{bmatrix} a & 0 & \sqrt{a}\end{bmatrix}^T : a \in R, a \geq 0\right\}$

$A = \begin{bmatrix} 2 & 0 & \sqrt{2}\end{bmatrix}^T \in W$

$B = \begin{bmatrix} 3 & 0 & \sqrt{3}\end{bmatrix}^T$

$A + B = \begin{bmatrix} 5 & 0 & \sqrt{2}+\sqrt{3}\end{bmatrix}$

$\sqrt{2}+\sqrt{3} \neq \sqrt{5}$!

$\boxed{\text{not closed under addition.}}$

Example 6: For the matrices

$A = \begin{bmatrix} 2 & -3 \\ 4 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 5 \\ 1 & -2 \end{bmatrix}$

$\vec{v}_1$ $\vec{v}_2$

in $M_{2,2}$, determine whether the given matrix is a linear combination of A and B.

$\begin{bmatrix} 6 & -19 \\ 10 & 7 \end{bmatrix}$

$\vec{z}$

$c_1\vec{v}_1 + c_2\vec{v}_2 = \vec{z}$

$c_1A + c_2B = \begin{bmatrix} 6 & -19 \\ 10 & 7 \end{bmatrix}$

$\begin{bmatrix} 2c_1 & -3c_1 \\ 4c_1 & 1c_1 \end{bmatrix} + \begin{bmatrix} 0c_2 & 5c_2 \\ 1c_2 & -2c_2 \end{bmatrix} = \begin{bmatrix} 6 & -19 \\ 10 & 7 \end{bmatrix}$

$$\left.\begin{aligned} 2c_1 &= 6 \\ -3c_1 + 5c_2 &= -19 \\ 4c_1 + c_2 &= 10 \\ c_1 - 2c_2 &= 7 \end{aligned}\right\} \quad \begin{aligned} c_1 &= 3 \\ -3(3) + 5c_2 &= -19 \\ c_2 &= -2 \end{aligned}$$

$$\boxed{3\begin{bmatrix} 2 & -3 \\ 4 & 1 \end{bmatrix} + (-2)\begin{bmatrix} 0 & 5 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} 6 & -19 \\ 10 & 7 \end{bmatrix} \text{ yes}}$$

Consider $P_n(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots + a_n x^n$

$P_2(x) = a_0 + a_1 x + a_2 x^2$

Example 7: Determine whether the set of vectors in P_2 is linearly independent or linearly dependent.

$S = \{x^2, x^2+1\}$

$\vec{v}_1, \vec{v}_2, \vec{v}_3$

$c_1 \vec{v}_1 + c_2 \vec{v}_2 = \vec{0}$

$c_1 x^2 + c_2(x_2+1) = 0 + 0x + 0x^2$

$c_2 + (c_1 + c_2) x^2 = 0 + 0x^2$

S is linearly independent

$c_2 = 0$

$c_1 + c_2 = 0 \rightarrow c_1 = 0$

Example 8: Determine whether the set of vectors in $M_{2,2}$ is linearly independent or linearly dependent.

$$S = \left\{ \begin{bmatrix} 2 & 0 \\ -3 & 1 \end{bmatrix}, \begin{bmatrix} -4 & -1 \\ 0 & 5 \end{bmatrix}, \begin{bmatrix} -8 & -3 \\ -6 & 17 \end{bmatrix} \right\}$$

$\vec{v}_1 \quad \vec{v}_2 \quad \vec{v}_3$

$c_1 \vec{v}_1 + c_2 \vec{v}_2 + c_3 \vec{v}_3 = \vec{0}$

$$c_1 \begin{bmatrix} 2 & 0 \\ -3 & 1 \end{bmatrix} + c_2 \begin{bmatrix} -4 & -1 \\ 0 & 5 \end{bmatrix} + c_3 \begin{bmatrix} -8 & -3 \\ -6 & +17 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$2c_1 - 4c_2 - 8c_3 = 0 \rightarrow 2(-2c_3) - 4(-3c_3) - 8c_3 = 0 \rightarrow c_3 = 1$

$-c_2 - 3c_3 = 0 \rightarrow c_2 = -3c_3 = -3$

$-3c_1 - 6c_3 = 0 \rightarrow c_1 = -2c_3 = -2$

$c_1 + 5c_2 - 17c_3 = 0$

$\boxed{\text{Since } \exists \text{ a nontrivial solution to this equation, S is linearly dependent.}}$

Example 9: Write the standard basis for the vector space.

a. $M_{3,2}$

Standard basis for $M_{3,2}$ = $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

$$\begin{cases} a_0 = 1: 1 + 0x + 0x^2 + 0x^3 \\ a_1 = 1: 0 + 1x + 0x^2 + 0x^3 \\ a_2 = 1: 0 + 0x + 1x^2 + 0x^3 \\ a_3 = 1: 0 + 0x + 0x^2 + 1x^3 \end{cases}$$

b. $P_3 = a_0 + a_1 x + a_2 x^2 + a_3 x^3$

Standard basis = $\{1, x, x^2, x^3\}$

Example 10: Determine whether S is a basis for the indicated vector space.

$S = \{\underset{\vec{v}_1}{4t - t^2}, \underset{\vec{v}_2}{5 + t^3}, \underset{\vec{v}_3}{3t + 5}, \underset{\vec{v}_4}{2t^3 - 3t^2}\}$ for P_3

$\text{Dim}(P_3) = 4$

Check for lin. ind:

$c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 + c_4\vec{v}_4 = \vec{0}$

$c_1(4t - t^2) + c_2(5 + t^3) + c_3(3t + 5) + c_4(2t^3 - 3t^2) = 0$

$4c_1t - c_1t^2 + 5c_2 + c_2t^3 + 3c_3t + 5c_3 + 2c_4t^3 - 3c_4t^2 = 0 + 0t + 0t^2 + 0t^3$

$(5c_2 + 5c_3) + (4c_1 + 3c_3)t + (-c_1 - 3c_4)t^2 + (c_2 + 2c_4) = 0 + 0t + 0t^2 + 0t^3$

$$\begin{cases} 5c_2 + 5c_3 = 0 \\ 4c_1 + 3c_3 = 0 \\ -c_1 - 3c_4 = 0 \\ c_2 + 2c_4 = 0 \end{cases} \quad A = \begin{bmatrix} 0 & 5 & 5 & 0 \\ 4 & 0 & 3 & 0 \\ -1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 2 \end{bmatrix}$$

$\det(A) = 30 \neq 0$ so $\exists$ a unique solution to the system, so S spans P_3.

Since S spans P_3, $\dim(P_3) = 4$, and S has 4 vectors, S is a basis for P_3.

Example 11: Find a basis for the vector space of all 3 x 3 symmetric matrices. What is the dimension of this vector space?

1) Hmmm... the easiest basis to find is the standard basis.

2) What does a 3x3 symmetric matrix look like in general?

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$$

Standard basis for 3x3 symmetric matrices:

$$\left\{ \begin{bmatrix} 1&0&0\\0&0&0\\0&0&0 \end{bmatrix}, \begin{bmatrix} 0&1&0\\1&0&0\\0&0&0 \end{bmatrix}, \begin{bmatrix} 0&0&1\\0&0&0\\1&0&0 \end{bmatrix}, \begin{bmatrix} 0&0&0\\0&1&0\\0&0&0 \end{bmatrix}, \begin{bmatrix} 0&0&0\\0&0&1\\0&1&0 \end{bmatrix}, \begin{bmatrix} 0&0&0\\0&0&0\\0&0&1 \end{bmatrix} \right\}$$

dimension for this space = 6

Example 11: Let T be the linear transformation from P_2 into R given by the integral $T(p)=\int_0^1 p(x)dx$.

Find the preimage of 1. That is, find the polynomial function(s) of degree 2 or less such that $T(p)=1$.

1) $T: P_2 \to R$

$T(P_2) = R$

$\int_0^1 p(x)dx = 1$

$\int_0^1 [a_0 + a_1 x + a_2 x^2] dx = 1$

$\left(a_0 x + \frac{1}{2}a_1 x^2 + \frac{1}{3}a_2 x^3\right)\Big|_{x=0}^{x=1} = 1$

$\left(a_0 + \frac{1}{2}a_1 + \frac{1}{3}a_2\right) - (0) = 1$

$a_0 = 1 - \frac{1}{2}a_1 - \frac{1}{3}a_2$, Let $a_1 = 2a$ and $a_2 = 3b$

$a_0 = 1 - a - b$

2) $P_2(x) = a_0 + a_1 x + a_2 x^2$

$$\boxed{P(x) = \{(1-a-b) + 2ax + 3bx^2 : a, b \in R\}}$$

3.4: RANK/NULLITY OF A MATRIX, SYSTEMS OF LINEAR EQUATIONS. AND COORDINATE VECTORS

Learning Objectives:

1. Find a basis for the row space, a basis for the column space, and the rank of a matrix
2. Find the nullspace of a matrix
3. Find a coordinate matrix relative to a basis in R^n
4. Find the transition matrix from the basis B to the basis B' in R^n
5. Represent coordinates in general *n*-dimensional spaces

Let's do our math stretches!

Consider the following matrix.

$$A = \begin{array}{c c} \begin{matrix} \vec{c}_1 & \vec{c}_2 & \vec{c}_3 & \vec{c}_4 \end{matrix} & \\ \begin{bmatrix} 1 & 3 & -1 & 5 \\ 7 & 1 & 13 & 6 \end{bmatrix} & \begin{matrix} \vec{r}_1 \\ \vec{r}_2 \end{matrix} \end{array}$$

The row vectors of A are:

$(1, 3, -1, 5), (7, 1, 13, 6)$

or $\begin{bmatrix} 1 & 3 & -1 & 5 \end{bmatrix}, \begin{bmatrix} 7 & 1 & 13 & 6 \end{bmatrix}$

The column vectors of A are:

$(1,7)^T, (3,1)^T, (-1,13)^T, (5,6)^T$

$\begin{bmatrix} 1 \\ 7 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 13 \end{bmatrix}, \begin{bmatrix} 5 \\ 6 \end{bmatrix}$

DEFINITION OF ROW SPACE AND COLUMN SPACE OF A MATRIX

Let A be an $m \times n$ matrix.

The row space of A is the subspace of R^n spanned by the row vectors of A.

The column space of A is the subspace of R^m spanned by the column vectors of A.

Recall that two matrices are row-equivalent when one can be obtained from the other by elementary row *operations.*

THEOREM 3.12: ROW-EQUIVALENT MATRICES HAVE THE SAME ROW SPACE

If an $m \times n$ matrix A is row-equivalent to an $m \times n$ matrix B, then the row space of A is equal to the row space of B.

Proof:

Since A is row-equivalent to B, ∃ a finite number of elementary matrices $E_1, E_2, \ldots, E_k \ni B = E_k E_{k-1} \cdots E_2 E_1 A$, it follows that the row vectors of B can be written as linear combinations of the row vectors of A. The row vectors of B lie in the row space of A, and the subspace spanned by the row vectors of B is contained in the row space of A. Similarly, the row vectors of A lie in the row space of B, and the subspace spanned by the row vectors of A is contained in the row space of B. ∴ The 2 rowspaces are subspaces of each other, hence they are equal.

THEOREM 3.12: BASIS FOR THE ROW SPACE OF A MATRIX

If a matrix A is row-equivalent to a matrix B in row-echelon form, then the nonzero row vectors of B form a basis for the row space of A.

To find a basis for the row space of a matrix: row reduce the matrix. The nonzero rows in the reduced matrix are a basis for the row space of the matrix. Your answer should be in the form of a set of row vectors.

To find a basis for the column space of a matrix:

Method 1: Use the steps above on the transpose of the matrix. Your answer should be in the form of a set of column vectors.

Method 2: Use reduced form of the original matrix to find the columns which contain the pivots (leading ones). Use the corresponding columns from the original matrix for a basis. Your answer should be in the form of a set of column vectors.

$$\begin{array}{cc} \vec{c}_1 & \vec{c}_2 & \\ \end{array}$$

Example 1: Find a basis for the row space and column space of the following matrix:

$$A = \begin{bmatrix} 2 & -3 & 1 \\ 5 & 10 & 6 \\ 8 & -7 & 5 \end{bmatrix}, \quad B = \text{rref}(A) = \begin{array}{ccc|c} \vec{c}_1 & \vec{c}_2 & \vec{c}_3 & \\ \hline 1 & 0 & 4/5 & \vec{r}_1 \\ 0 & 1 & 1/5 & \vec{r}_2 \\ 0 & 0 & 0 & \vec{r}_3 \end{array}$$

$$\boxed{\text{A Basis for the row space: } \{(1,0,4/5),(0,1,1/5)\}}$$

Method 2:

$$\boxed{\text{A Basis for the column space: } \left\{\begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix}, \begin{bmatrix} -3 \\ 10 \\ -7 \end{bmatrix}\right\}}$$

Method 1:

$$\text{rref}(A^T) = \begin{bmatrix} 1 & 0 & 23/7 \\ 0 & 1 & 2/7 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\boxed{\text{A basis for the column space: } \left\{\begin{bmatrix} 1 \\ 0 \\ \frac{23}{7} \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ \frac{2}{7} \end{bmatrix}\right\}}$$

Example 2: Find a basis for the row space and column space of the following matrix:

$$A = \begin{bmatrix} 4 & 20 & 31 \\ 6 & -5 & -6 \\ 2 & -11 & -16 \end{bmatrix} \qquad \text{rref}(A) = \begin{array}{ccc|c} 1 & 0 & 1/4 & \vec{r}_1 \\ 0 & 1 & 3/2 & \vec{r}_2 \\ 0 & 0 & 0 & \vec{r}_3 \\ \hline \vec{c}_1 & \vec{c}_2 & \vec{c}_3 & \end{array}$$

$$\boxed{\text{A basis for the row space: } \{(1,0,1/4),(0,1,3/2)\}}$$

$$\boxed{\text{A basis for the column space: } \left\{\begin{bmatrix} 4 \\ 6 \\ 2 \end{bmatrix}, \begin{bmatrix} 20 \\ -5 \\ -11 \end{bmatrix}\right\}}$$

THEOREM 3.13: ROW AND COLUMN SPACES HAVE EQUAL DIMENSIONS

If A is an $m \times n$ matrix, then the row space and the column space of A have the same dimension.

DEFINITION OF THE RANK OF A MATRIX

The dimension of the row (or column) space of a matrix A is called the rank of A and is denoted by rank(A).

Example 3: Find the rank of the matrix from

a. Example 1

$\boxed{\text{rank}(A) = 2}$

b. Example 2

$\boxed{\text{rank}(A) = 2}$

THEOREM 3.14: SOLUTIONS OF A HOMOGENEOUS SYSTEM

If A is an $m \times n$ matrix, then the set of all solutions of the homogeneous system of linear equations $A\vec{x} = \vec{0}$ is a subspace of R^n called the nullspace of A and is denoted $N(A)$. So,

$$N(A) = \{\vec{x} \in R^n : A\vec{x} = \vec{0}\}$$

The dimension of the nullspace of A is called the nullity of A.

Proof:

Since A is $m \times n$, $\vec{x}$ has size $n \times 1$. So the set of all solutions has to be a subset of R^n. This set has to be nonempty since $A\vec{0} = \vec{0}$.

$A(\vec{x}_1 + \vec{x}_2) = A\vec{x}_1 + A\vec{x}_2 = \vec{0} + \vec{0} = \vec{0}$, so A is closed under +.

$A(c\vec{x}_1) = c(A\vec{x}_1) = c\vec{0} = \vec{0}$, so A is closed under scal. mult.

$\therefore$ $A\vec{x} = \vec{0}$ is a subspace of R^n. //

Example 4: Find the nullspace of the following matrix A, and determine the nullity of A.

$$A = \begin{bmatrix} 1 & 4 & 2 & 1 \\ 0 & 1 & 1 & -1 \\ -2 & -8 & -4 & -2 \end{bmatrix}$$

3×4

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

$$A\vec{x} = \vec{0}$$

$$\text{rref}(A) = \begin{bmatrix} 1 & 0 & -2 & 5 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{matrix} x_1 - 2x_3 + 5x_4 = 0 \\ x_2 + x_3 - x_4 = 0 \end{matrix}$$

$x_1 = 2s - 5t \quad x_3 = s$

$x_2 = -s + t \quad x_4 = t$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2s-5t \\ -s+t \\ s \\ t \end{bmatrix} = \begin{bmatrix} 2s \\ -s \\ s \\ 0 \end{bmatrix} + \begin{bmatrix} -5t \\ t \\ 0 \\ t \end{bmatrix} = s\begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -5 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

$$N(A) = \{(2s-5t, -s+t, s, t) : s,t \in \mathbb{R}\}$$
$$= \{s(2,-1,1,0) + t(-5,1,0,1) : s,t \in \mathbb{R}\}$$

$$\boxed{\text{nullity}(A) = 2}$$

A basis for the $N(A)$: $\left\{ \begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}$

THEOREM 3.15: DIMENSION OF THE SOLUTION SPACE

If A is an $m \times n$ matrix of rank __r__, then the __dimension__ of the solution space of __$A\vec{x} = \vec{0}$__ is __$n - r$__. That is,

$$n = \text{rank}(A) + \text{nullity}(A)$$

Example 5: consider the following homogeneous system of linear equations:

$$x - y = 0$$
$$-x + y = 0$$

$\rightarrow$ homogeneous

a. Find a basis for the solution space.

$$A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \quad \vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\text{ref}(A) = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \rightarrow x - y = 0 \rightarrow x = y \rightarrow x = t, \; y = t$$

$$\vec{x} = \begin{bmatrix} t \\ t \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\boxed{\text{A basis for the solution space is: } \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}}$$

b. Find the dimension of the solution space. (nullity(A))

$$\boxed{1}$$

c. Find the solution of a consistent system $A\mathbf{x} = \mathbf{b}$ in the form $\mathbf{x}_p + \mathbf{x}_h$

$$A\vec{x} = \vec{b} \rightarrow \vec{x} = \underbrace{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}_{\vec{x}_p} + \underbrace{t\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{\vec{x}_h}$$

$$\vec{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

THEOREM 3.16: SOLUTIONS OF A NONHOMOGENEOUS LINEAR SYSTEM

If $\mathbf{x}_p$ is a particular solution of the nonhomogeneous system $A\mathbf{x} = \mathbf{b}$, then every solution of this system can be written in the form $\vec{x} = \vec{x}_p + \vec{x}_h$ where $\mathbf{x}_h$ is a solution of the corresponding homogeneous system $A\vec{x} = \vec{0}$.

Proof: Let $\vec{x}$ be any solution of $A\vec{x} = \vec{b}$. Then $\vec{x} - \vec{x}_p$ is a solution to $A\vec{x} = \vec{0}$. $A(\vec{x} - \vec{x}_p) = \vec{0} \rightarrow A\vec{x} - A\vec{x}_p = \vec{0}$, which gives us $\vec{b} - \vec{b} = \vec{0}$. Let $\vec{x}_h = \vec{x} - \vec{x}_p$, thus $\vec{x} = \vec{x}_p + \vec{x}_h$. //

THEOREM 3.17: SOLUTIONS OF A SYSTEM OF LINEAR EQUATIONS

The system $A\vec{x} = \vec{b}$ is consistent if and only if $\vec{b}$ is in the column space of A.

Proof:

$$A\vec{x} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ a_{3n} \\ \vdots \\ a_{mn} \end{bmatrix},$$

So $A\vec{x} = \vec{b}$ iff $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$ is a linear combo of the columns of A. That is, the system is consistent iff $\vec{b} \in$ subspace R^m spanned by the columns of A. //

Example 7: consider the following nonhomogeneous system of linear equations:

$$\begin{aligned} 2x-4y\ +5z &= 8 \\ -7x+14y+4z &= -28 \\ 3x\ -6y\ \ +z &= 12 \end{aligned}$$

$$\vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Determine whether $A\mathbf{x}=\mathbf{b}$ is consistent.

$$\left[\begin{array}{ccc|c} 2 & -4 & 5 & 8 \\ -7 & 14 & 4 & -28 \\ 3 & -6 & 1 & 12 \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{ccc|c} 1 & -2 & 0 & 4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \rightarrow \begin{aligned} &x-2y=4 \rightarrow x=2t+4 \\ &z=0 \\ &y=t \end{aligned}$$

$$\vec{x} = \begin{bmatrix} 2t+4 \\ t \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 2t \\ t \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

So $A\vec{x}=\vec{b}$ is consistent.

If the system is consistent, write the solution in the form $\mathbf{x}=\mathbf{x}_p+\mathbf{x}_h$, where $\mathbf{x}_p$ is a particular solution of $A\mathbf{x}=\mathbf{b}$ and $\mathbf{x}_h$ is a solution of $A\mathbf{x}=\mathbf{0}$.

$$\boxed{\vec{x} = \underbrace{\begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix}}_{\vec{x}_p} + \underbrace{\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}}_{\vec{x}_h} \text{ is a solution.}}$$

COORDINATE REPRESENTATION RELATIVE TO A BASIS

Let $B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be an ordered basis for a vector space V, and let $\mathbf{x}$ be a vector in V such that

$$\vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n$$

The scalars $c_1, c_2, \ldots, c_n$ are called the coordinates of $\vec{x}$ relative to the basis B. The column matrix (or coordinate matrix) of $\vec{x}$ relative to B is the column matrix in R^n whose components are the coordinates of $\vec{x}$.

$$[\vec{x}]_B = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

Note: In R^n, column notation is used for the coordinate matrix. For the vector $\vec{x} = (x_1, x_2, \ldots, x_n)$, the x_i's are the coordinates of $\vec{x}$ (relative to the standard basis S for R^n. So you have

$$[\vec{x}]_S = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Example 8: Find the coordinate matrix of $\mathbf{x}$ in R^n relative to the standard basis.

$\mathbf{x} = (1, -3, 0)$

$$S = \{(1,0,0), (0,1,0), (0,0,1)\}$$

$$\vec{x} = 1(1,0,0) - 3(0,1,0) + 0(0,0,1)$$

$$\boxed{[\vec{x}]_S = \begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix}}$$

Example 9: Given the coordinate matrix of $\mathbf{x}$ relative to a (nonstandard) basis B for R^n, find the coordinate matrix of $\mathbf{x}$ relative to the standard basis.

$$B = \{(4,0,7,3),(0,5,-1,-1),(-3,4,2,1),(0,1,5,0)\}$$

$$\underset{\vec{v}_1}{} \quad \underset{\vec{v}_2}{} \quad \underset{\vec{v}_3}{} \quad \underset{\vec{v}_4}{}$$

$$[\mathbf{x}]_B = \begin{bmatrix} -2 \\ 3 \\ 4 \\ 1 \end{bmatrix}$$

$$\vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 + c_4\vec{v}_4$$

$$\vec{x} = -2(4,0,7,3) + 3(0,5,-1,-1) + 4(-3,4,2,1) + 1(0,1,5,0)$$

$$\boxed{\vec{x} = (-20, 32, -4, -5)}$$

Example 10: Find coordinate matrix of $\mathbf{x}$ in R^n relative to the basis B'.

$B' = \{(-6,7),(4,-3)\}$, $\mathbf{x} = (-26,32)$

$$\underset{\vec{v}_1}{} \quad \underset{\vec{v}_2}{}$$

$$\vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2$$

$$(-26,32) = c_1(-6,7) + c_2(4,-3)$$

$$-6c_1 + 4c_2 = -26$$

$$7c_1 - 3c_2 = 32$$

$$c_1 = 5, \; c_2 = 1$$

$$\boxed{[\vec{x}]_{B'} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}}$$

The matrix P is called the transition matrix from B' to B, where $[\vec{x}]_{B'}$ is the coordinate matrix of $\vec{x}$ relative to B', and $[\vec{x}]_B$ is the coordinate matrix of $\vec{x}$ relative to B.

Multiplication by the transition matrix P changes a coordinate matrix relative to B' into a coordinate matrix relative to B.

Change of basis from B' to B:

$$P[\vec{x}]_{B'} = [\vec{x}]_B$$

Change of basis from B to B':

$$P^{-1}[\vec{x}]_B = [\vec{x}]_{B'}$$

The change of basis problem in example 10 can be represented by the matrix equation:

$$-6c_1 + 4c_2 = -26$$
$$7c_1 - 3c_2 = 32$$

$$P = \begin{bmatrix} -6 & 4 \\ 7 & -3 \end{bmatrix}, \quad [\vec{x}]_S = \begin{bmatrix} -26 \\ 32 \end{bmatrix}$$

$$P[\vec{x}]_{B'} = [x]_S$$

$$[\vec{x}]_{B'} = P^{-1}\begin{bmatrix} -26 \\ 32 \end{bmatrix} = -\frac{1}{10}\begin{bmatrix} -3 & -4 \\ -7 & -6 \end{bmatrix}\begin{bmatrix} -26 \\ 32 \end{bmatrix} = -\frac{1}{10}\begin{bmatrix} -50 \\ -10 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \end{bmatrix} \checkmark$$

THEOREM 3.18: THE INVERSE OF A TRANSITION MATRIX

If P is the transition matrix from a basis B' to a basis B in R^n, then P is invertible and the transition matrix from B to B' is given by P^{-1}.

FYI: The transition matrix from B' to B is P.

LEMMA

Let $B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ and $B' = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ be two bases for a vector space V. If

$$\mathbf{v}_1 = c_{11}\mathbf{u}_1 + c_{21}\mathbf{u}_2 + \cdots c_{n1}\mathbf{u}_n$$
$$\mathbf{v}_2 = c_{12}\mathbf{u}_1 + c_{22}\mathbf{u}_2 + \cdots c_{n2}\mathbf{u}_n$$
$$\vdots$$
$$\mathbf{v}_n = c_{1n}\mathbf{u}_1 + c_{2n}\mathbf{u}_2 + \cdots c_{nn}\mathbf{u}_n$$

then the transition matrix from B to B' is

$$Q = \begin{bmatrix} c_{11} & \cdots & c_{1n} \\ \vdots & \ddots & \vdots \\ c_{n1} & \cdots & c_{nn} \end{bmatrix}$$

THEOREM 3.19: TRANSITION MATRIX FROM B TO B'

Let $B=\{\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_n\}$ and $B'=\{\mathbf{u}_1,\mathbf{u}_2,\ldots,\mathbf{u}_n\}$ be two bases for R^n. Then the transition matrix P^{-1} from B to B' can be found using Gauss-Jordan elimination on the $n\times 2n$ matrix $[B' \quad B]$ as follows.

row reduce $[B' \ B]$ to $[I_n \ P^{-1}]$

Note: The transition matrix from B' to B can be found using Gauss-Jordan elimination on the $n\times 2n$ matrix $[B \ B']$ as follows.

row reduce $[B \ B']$ to $[I_n \ P]$

Example 11: Find the transition matrix from B to B'.

$B=\{(1,1),(1,0)\}$, $\underbrace{B'=\{(1,0),(0,1)\}}_{\text{standard basis for } R^2}$

$$[B' \ B]=\left[\begin{array}{cc|cc} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{array}\right]$$

I_n $\quad$ P^{-1}

$$\boxed{P^{-1}=\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}}$$

Example 12: Find the coordinate matrix of p relative to the standard basis for P_3.

$p=3x^2+114x+13$

$S=\{1, x, x^2, x^3\}$

$\vec{v}_1 \ \vec{v}_2 \ \vec{v}_3 \ \vec{v}_4$

$c_1(1)+c_2(x)+c_3(x^2)+c_4(x^3)=13+114x+3x^2$

$c_1+c_2x+c_3x^2+c_4x^3=13+114x+3x^2+0x^3$

$c_1=13$

$c_2=114$

$c_3=3$

$c_4=0$

$$\boxed{[\vec{p}]_S=\begin{bmatrix} 13 \\ 114 \\ 3 \\ 0 \end{bmatrix}}$$

3.5: THE KERNEL, RANGE, AND MATRIX REPRESENTATIONS OF LINEAR TRANSFORMATIONS, AND SIMILAR MATRICES

Learning Objectives:

1. Find the kernel of a linear transformation
2. Find a basis for the range, the rank, and the nullity of a linear transformation
3. Determine whether a linear transformation is one-to-one or onto
4. Determine whether vector spaces are isomorphic
5. Find the standard matrix for a linear transformation
6. Find the standard matrix for the composition of linear transformations and find the inverse of an invertible linear transformation
7. Find the matrix for a linear transformation relative to a nonstandard basis
8. Find and use a matrix for a linear transformation
9. Show that two matrices are similar and use the properties of similar matrices

THE KERNEL OF A LINEAR TRANSFORMATION

We know from an earlier theorem that for any linear transformation ______________, the zero vector in ____ maps to the __________ vector in _____. That is, _____________. In this section, we will consider whether there are other vectors ____ such that ______________. The collection of all such __________________ is called the _______________ of _____. Note that the zero vector is denoted by the symbol_____ in both _____ and _____, even though these two zero vectors are often different.

DEFINITION OF KERNEL OF A LINEAR TRANSFORMATION

Let $T : V \to W$ be a linear transformation. Then the set of all vectors $\mathbf{v}$ in V that satisfy _______________ is called the ________________ of T and is denoted by _________________.

Example 1: Find the kernel of the linear transformation.

a. $T : R^3 \to R^3, T(x, y, z) = (x, 0, z)$

b. $T: P_3 \to P_2, T\left(a_0 + a_1x + a_2x^2 + a_3x^3\right) = a_1 + 2a_2x + 3a_3x^2$

c.

$T: P_2 \to R,$

$T(p) = \int_0^1 p(x)dx$

THEOREM 3.20: THE KERNEL IS A SUBSPACE OF V

The kernel of a linear transformation $T: V \to W$ is a subspace of the domain V.

Proof:

THEOREM 3.20: COROLLARY

Let $T : R^n \to R^m$ be the linear transformation given by $T(\mathbf{x}) = A\mathbf{x}$. Then the kernel of T is equal to the solution space of ____________________.

THEOREM 3.21: THE RANGE OF T IS A SUBSPACE OF W

The range of a linear transformation $T : V \to W$ is a subspace of W.

THEOREM 3.21: COROLLARY

Let $T : R^n \to R^m$ be the linear transformation given by $T(\mathbf{x}) = A\mathbf{x}$. Then the column space of ________ is equal to the _______________ of _____.

Example 2: Let $T(\mathbf{v}) = A\mathbf{v}$ represent the linear transformation T. Find a basis for the kernel of T and the range of T.

$$A = \begin{bmatrix} 1 & 1 \\ -1 & 2 \\ 0 & 1 \end{bmatrix}$$

DEFINITION OF RANK AND NULLITY OF A LINEAR TRANSFORMATION

Let $T:V\to W$ be a linear transformation. The dimension of the kernel of T is called the

____________________ of T and is denoted by ______________________. The dimension of the range of T

is called the ______________ of T and is denoted by ____________________.

THEOREM 3.22: SUM OF RANK AND NULLITY

Let $T:V\to W$ be a linear transformation from an *n*-dimensional vector space V into a vector space W. Then

the ________ of the ______________________ of the _______________ and ______________________ is

equal to the dimension of the ___________________. That is,

Proof:

Example 3: Define the linear transformation T by $T(\mathbf{x}) = A\mathbf{x}$. Find $\ker(T)$, $\text{null}(T)$, $\text{range}(T)$, and $\text{rank}(T)$.

$$A = \begin{bmatrix} 3 & -2 & 6 & -1 & 15 \\ 4 & 3 & 8 & 10 & -14 \\ 2 & -3 & 4 & -4 & 20 \end{bmatrix}$$

Example 4: Let $T: R^3 \to R^3$ be a linear transformation. Use the given information to find the nullity of T and give a geometric description of the kernel and range of T.

T is the reflection through the *yz*-coordinate plane:

$$T(x, y, z) = (-x, y, z)$$

ONE-TO-ONE AND ONTO LINEAR TRANSFORMATIONS

If the _______________ vector is the only vector ____ such that _____________, then ____ is _______________________. A function _______________ is called one-to-one when the __________________ of every _____ in the range consists of a _________________ vector. This is equivalent to saying that ____ is one-to-one if and only if, for all ____ and ____ in _____, ____________________ implies that _______________.

THEOREM 3.23: ONE-TO-ONE LINEAR TRANSFORMATIONS

Let $T: V \rightarrow W$ be a linear transformation. Then T is one-to-one if and only if ________________________.

Proof:

THEOREM 3.24: ONTO LINEAR TRANSFORMATIONS

Let $T: V \to W$ be a linear transformation, where W is finite dimensional. Then T is onto if and only if the ____________ of T is equal to the ______________________ of W.

Proof:

THEOREM 3.25: ONE-TO-ONE AND ONTO LINEAR TRANSFORMATIONS

Let $T: V \to W$ be a linear transformation with vector spaces V and W, ___________ of dimension *n.* Then T is one-to-one if and only if it is ________.

Example 5: Determine whether the linear transformation is one-to-one, onto, or neither.

$T: R^2 \to R^2, T(x, y) = (x-y, y-x)$

DEFINITION: ISOMORPHISM

A linear transformation $T: V \to W$ that is _______________________ and ____________ is called an _________________________. Moreover, if V and W are vector spaces such that there exists an isomorphism from V to W, then V and W are said to be ________________________ to each other.

THEOREM 3.26: ISOMORPHIC SPACES AND DIMENSION

Two finite dimensional vector spaces V and W are ________________________if and only if they are of the same _________________________.

Example 6: Determine a relationship among *m, n, j,* and *k* such that $M_{m,n}$ is isomorphic to $M_{j,k}$.

WHICH FORMAT IS BETTER? WHY?

Consider $T: R^3 \to R^3, T(x_1, x_2, x_3) = (4x_1 - x_2 - 5x_3, -2x_1 + x_2 + 6x_3, x_2 - 3x_3)$

and

$$T(\mathbf{x}) = A\mathbf{x} = \begin{bmatrix} 4 & -1 & -5 \\ -2 & 1 & 6 \\ 0 & 1 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

What do you think?

The key to representing a linear transformation ______________ by a matrix is to determine how it acts on a _______________ for _____. Once you know the _______________ of every vector in the ______________, you can use the properties of linear transformations to determine _______ for any ___ in ____.

Do you remember the standard basis for R^n? Write this standard basis for R^n in column vector notation.

$B = \{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\} =$

THEOREM 3.26: STANDARD MATRIX FOR A LINEAR TRANSFORMATION

Let $T : R^n \to R^m$ be a linear transformation such that, for the standard basis vectors $\mathbf{e}_i$ of R^n,

$$T(\mathbf{e}_1) = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix},\ T(\mathbf{e}_2) = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix},\ \ldots,\ T(\mathbf{e}_n) = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix},$$

then the $m \times n$ matrix whose n columns correspond to $T(\mathbf{e}_i)$

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}$$

is such that $T(\mathbf{v}) = A\mathbf{v}$ for every $\mathbf{v}$ in R^n. A is called the standard matrix for T.

Example 5: Find the standard matrix for the linear transformation T.

$$T(x, y) = (4x + y, 0, 2x - 3y)$$

Example 2: Use the standard matrix for the linear transformation T to find the image of the vector $\mathbf{v}$.

$$T(x, y) = (x + y, x - y, 2x, 2y),\ \mathbf{v} = (3, -3)$$

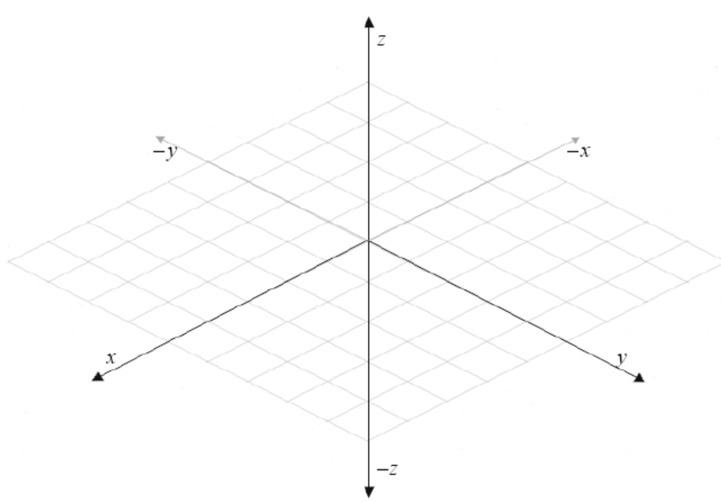
Example 6: Consider the following linear transformation T:

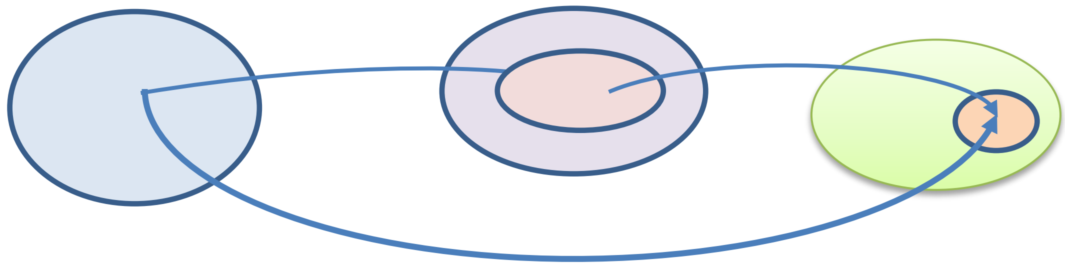
T is the reflection through the *yz*-coordinate plane in $R^3 : T(x,y,z) = (-x,y,z)$, $\mathbf{v} = (2,3,4)$.

a. Find the standard matrix A for the following linear transformation T.

b. Use A to find the image of the vector $\mathbf{v}$.

c. Sketch the graph of $\mathbf{v}$ and its image.





THEOREM 3.27: COMPOSITION OF LINEAR TRANSFORMATIONS

Let $T_1 : R^n \to R^m$ and $T_2 : R^m \to R^p$ be linear transformations with standard matrices A_1 and A_2, respectively. The composition $T : R^n \to R^p$, defined by $T(\mathbf{v}) = T_2(T_1(\mathbf{v}))$, is a linear transformation. Moreover, the standard matrix A for T is given by the matrix product $A = A_2 A_1$.

Proof:

Example 7: Find the standard matrices A and A' for $T = T_2 \circ T_1$ and $T = T_1 \circ T_2$.

$T_1 : R^2 \to R^3,\ T_1(x, y) = (x, y, y)$

$T_2 : R^3 \to R^2,\ T_2(x, y, z) = (y, z)$

DEFINITION OF INVERSE LINEAR TRANSFORMATION

If $T_1 : R^n \to R^n$ and $T_2 : R^n \to R^n$ are linear transformations such that for every $\mathbf{v}$ in R^n,

then T_2 is called the ___________________ of T_1, and T_1 is said to be ___________________.

**Not every ______________ transformation has an _________________. If _____ is ___________________,

however, the inverse is _________________ and is denoted by _______________.

THEOREM 3.28

Let is $T : R^n \to R^n$ be a linear transformation with a standard matrix A. Then the following conditions are equivalent.

1. T is ______________________.

2. T is an ______________________________.

3. A is ________________________.

4. If T is invertible with standard matrix A, then the standard matrix for ________ is __________.

Example 8: Determine whether the linear transformation $T(x, y) = (x + y, x - y)$ is invertible. If it is, find its inverse.

THEOREM 3.29: TRANSFORMATION MATRIX FOR NONSTANDARD BASES

Let V and W be finite-dimensional vector spaces with bases B and B', respectively, where $B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$.

If $T : V \to W$ is a linear transformation such that

$$\left[T(\mathbf{v}_1)\right]_{B'} = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}, \ \left[T(\mathbf{v}_2)\right]_{B'} = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}, \ \ldots, \ \left[T(\mathbf{v}_n)\right]_{B'} = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix},$$

then the $m \times n$ matrix whose n columns correspond to $\left[T(\mathbf{v}_1)\right]_{B'}$

$$A = \begin{bmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & \vdots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}$$

Example 9: Find $T(\mathbf{v})$ by using (a) the standard matrix, and (b) the matrix relative to B and B'.

$T : R^3 \to R^2$, $T(x, y, z) = (x - y, y - z)$, $\mathbf{v} = (1, 2, 3)$,

$B = \{(1,1,1), (1,1,0), (0,1,1)\}$, $B' = \{(1,2), (1,1)\}$

Example 10: Let $B=\left\{e^{2x}, xe^{2x}, x^2e^{2x}\right\}$ be a basis for a subspace of W of the space of continuous functions, and let D_x be the differential operator on W. Find the matrix for D_x relative to the basis B.

A classical problem in linear algebra is determining whether it is possible to find a basis _____ such that the

matrix for _____ relative to _____ is ______________.

1. Matrix for T relative to B : ________
2. Matrix for T relative to B' : ________
3. Transition matrix from B' to B : ________
4. Transition matrix from B to B' : ________

Example 11: Find the matrix A' relative to the basis B'.

$T: R^2 \to R^2,\ T(x,y) = (x - 2y, 4x),\ B' = \{(-2,1),(-1,1)\}$

Example 12: Let $B=\{(1,-1),(-2,1)\}$ and $B'=\{(-1,1),(1,2)\}$ be bases for R^2, $[\mathbf{v}]_{B'}=[1 \quad -4]^T$, and let $A=\begin{bmatrix}2 & 1\\0 & -1\end{bmatrix}$ be the matrix for $T:R^2\to R^2$ relative to B.

a. Find the transition matrix P from B' to B.

b. Use the matrices P and A to find $[\mathbf{v}]_B$ and $\left[T(\mathbf{v})_{B'}\right]$ where $[\mathbf{v}]_{B'}=[1 \quad -4]^T$.

DEFINITION OF SIMILAR MATRICES

For square matrices A and A' of order n, A' is said to be similar to A when there exists an invertible matrix P such that $A' = P^{-1}AP$.

THEOREM 3.30

Let A, B, and C be square matrices of order n. Then the following properties are true.

1. A is ______________________ to ______.
2. If A is similar to B, then _____ is _______________ to ____.
3. If A is similar to B and B is similar to C, then _____ is __________________to _____.

Proof:

Example 13: Use the matrix P to show that A and A' are similar.

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}, A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}, A' = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & 2 & 3 \end{bmatrix}$$

DIAGONAL MATRICES

Diagonal matrices have many ___________________________ advantages over nondiagonal matrices.

$$D=\begin{pmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{pmatrix} \qquad D^k=\begin{pmatrix} \_\_ & 0 & \cdots & 0 \\ 0 & \_\_ & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \_\_ \end{pmatrix}$$

Also, a diagonal matrix is its own _____________________. Additionally, if all the diagonal elements are nonzero, then the inverse of a diagonal matrix is the matrix whose main diagonal elements are the ____________________ of corresponding elements in the original matrix. Because of these advantages, it is important to find ways (if possible) to choose a basis for _____ such that the ___________________________ matrix is ___________________.

Example 14: Suppose $A=\begin{bmatrix} \frac{3}{2} & -1 & -\frac{1}{2} \\ -\frac{1}{2} & 2 & \frac{1}{2} \\ \frac{1}{2} & 1 & \frac{5}{2} \end{bmatrix}$ is the matrix for $T:R^3 \rightarrow R^3$ relative to the standard basis.

Find the diagonal matrix A' for T relative to the basis $B'=\{(1,1,-1),(1,-1,1),(-1,1,1)\}$.

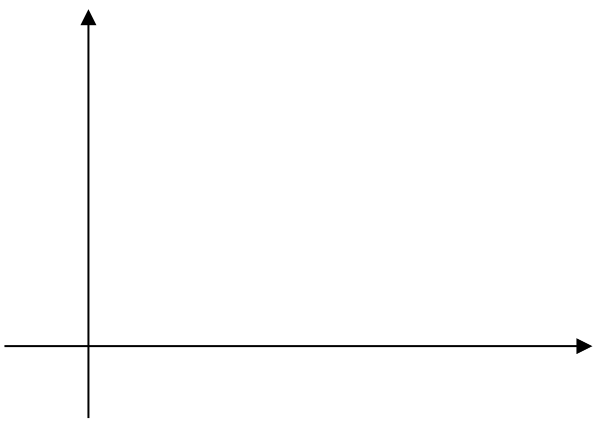
Example 15: Prove that if A is idempotent and B is similar to A, then B is idempotent. (An $n \times n$ matrix is idempotent when $A = A^2$).

Proof:

4.1: INNER PRODUCT SPACES

Learning Objectives:

1. Find the length of a vector and find a unit vector
2. Find the distance between two vectors
3. Find a dot product and the angle between two vectors, determine orthogonality, and verify the Cauchy-Schwartz Inequality, the triangle inequality, and the Pythagorean Theorem
4. Use a matrix product to represent a dot product
5. Determine whether a function defines an inner product, and find the inner product of two vectors in R^n, $M_{m,n}$, P_n, and $C[a,b]$
6. Find an orthogonal projection of a vector onto another vector in an inner product space



DEFINITION OF LENGTH OF A VECTOR IN R^n

The ____________________, _________________, or ____________________________ of a vector $\mathbf{v} = \{v_1, v_2, \ldots, v_n\}$

in ______ is given by

When would the length of a vector equal to 0?

Example 1: Consider the following vectors:

$\mathbf{u} = \left(1, \frac{1}{2}\right)$ $\mathbf{v} = \left(2, -\frac{1}{2}\right)$

a. Find $\|\mathbf{u}\|$

b. Find $\|\mathbf{v}\|$

c. Find $\|\mathbf{u}\|+\|\mathbf{v}\|$

d. Find $\|\mathbf{u}+\mathbf{v}\|$

e. Find $\|3\mathbf{u}\|$

f. Find $3\|\mathbf{u}\|$

Any observations?

THEOREM 4.1: LENGTH OF A SCALAR MULTIPLE

Let $\mathbf{v}$ be a vector in R^n and let c be a scalar. Then

where _____ is the _____________ ______________ of c.

Proof:

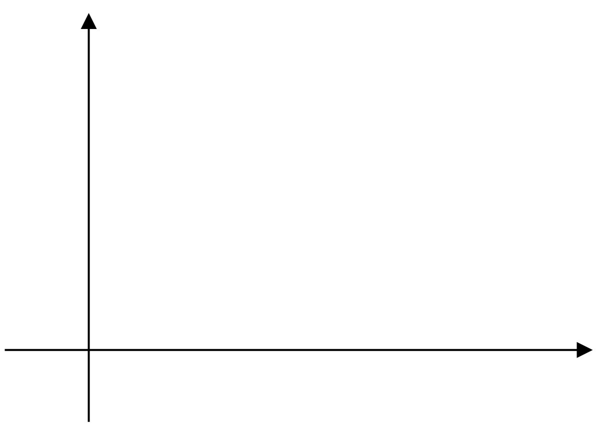
THEOREM 4.2: UNIT VECTOR IN THE DIRECTION OF $\mathbf{v}$

If $\mathbf{v}$ is a nonzero vector in R^n, then the vector

has length _____ and has the same ________________ as $\mathbf{v}$.

Proof:

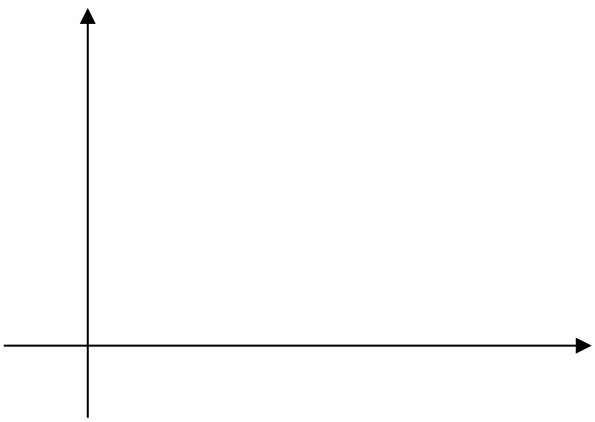
Example 2: Find the vector $\mathbf{v}$ with $\|\mathbf{v}\| = 3$ and the same direction as $\mathbf{u} = (0, 2, 1, -1)$.



DEFINITION OF DISTANCE BETWEEN TWO VECTORS

The distance between two vectors $\mathbf{u}$ and $\mathbf{v}$ in R^n is

Example 3: Find the distance between $\mathbf{u} = (1,1,2)$ and $\mathbf{v} = (-1,3,0)$.



DEFINITION OF DOT PRODUCT IN R^n

The dot product of $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ and $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ is the ______________ quantity

DEFINITION OF THE ANGLE BETWEEN TWO VECTORS IN R^n

The _____________ _____ between two nonzero vectors in R^n is given by

Example 4: Find the angle between $\mathbf{u} = (2, -1, 1)$ and $\mathbf{v} = (3, 0, 1)$.

Example 5: Consider the following vectors:

$\mathbf{u} = (-1, 2)$ $\mathbf{v} = (2, -2)$

a. Find $\mathbf{u} \cdot \mathbf{v}$

b. Find $\mathbf{v} \cdot \mathbf{v}$

c. Find $\|\mathbf{u}\|^2$

d. Find $(\mathbf{u} \cdot \mathbf{v})\mathbf{v}$

e. Find $\mathbf{u} \cdot (5\mathbf{v})$

THEOREM 4.3: PROPERTIES OF THE DOT PRODUCT

If $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are vectors in R^n, and c is a scalar, then the following properties are true.

1. $\mathbf{u}\cdot\mathbf{v} =$ ______________________________________
2. $\mathbf{u}\cdot(\mathbf{v}+\mathbf{w}) =$ ______________________________________
3. $c(\mathbf{u}\cdot\mathbf{v}) =$ __________________ = ____________________
4. $\mathbf{v}\cdot\mathbf{v} =$ __________________
5. $\mathbf{v}\cdot\mathbf{v} \geq 0$ and $\mathbf{v}\cdot\mathbf{v} = 0$ iff ________________________.

Example 6: Find $(3\mathbf{u}-\mathbf{v})\cdot(\mathbf{u}-3\mathbf{v})$ given that $\mathbf{u}\cdot\mathbf{u} = 8$, $\mathbf{u}\cdot\mathbf{v} = 7$, and $\mathbf{v}\cdot\mathbf{v} = 6$.

THEOREM 4.4: THE CAUCHY-SCWARZ INEQUALITY

If $\mathbf{u}$ and $\mathbf{v}$ are vectors in R^n, then

where ____________ denotes the _______________ value of $\mathbf{u}\cdot\mathbf{v}$.

Proof:

Example 7: Verify the Cauch-Schwarz Inequality for $\mathbf{u}=(-1,0)$ and $\mathbf{v}=(1,1)$.

DEFINITION OF ORTHOGONAL VECTORS

Two vectors $\mathbf{u}$ and $\mathbf{v}$ in R^n are orthogonal if

Example 7: Determine all vectors in R^2 that are orthogonal to $\mathbf{u} = (3,1)$.

THEOREM 4.5: THE TRIANGLE INEQUALITY

If $\mathbf{u}$ and $\mathbf{v}$ are vectors in R^n, then

Proof:

THEOREM 4.6: THE PYTHAGOREAN THEOREM

If $\mathbf{u}$ and $\mathbf{v}$ are vectors in R^n, then $\mathbf{u}$ and $\mathbf{v}$ are orthogonal if and only if

Example 8: Verify the Pythagoren Theorem for the vectors $\mathbf{u}=(3,-2)$ and $\mathbf{v}=(4,6)$.

DEFINITION OF AN INNER PRODUCT

Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors in a vector space V, and let c be any scalar. An inner product on V is a function that associates a real number $\langle \mathbf{u}, \mathbf{v} \rangle$ with each pair of vectors $\mathbf{u}$ and $\mathbf{v}$ and satisfies the following axioms.

1. $\langle \mathbf{u}, \mathbf{v} \rangle =$ ____________________
2. $\langle \mathbf{u}, \mathbf{v}+\mathbf{w} \rangle =$ ____________________________
3. $c\langle \mathbf{u}, \mathbf{v} \rangle =$ ____________________
4. $\langle \mathbf{v}, \mathbf{v} \rangle \geq 0$, and $\langle \mathbf{v}, \mathbf{v} \rangle = 0$ iff _______________

NOTE: The __________ product is the ______________________ ________________ product for _______.

Example 8: Show that the function $\langle \mathbf{u}, \mathbf{v} \rangle = u_1v_1 + 2u_2v_2 + u_3v_3$ defines an inner product on R^3, where , $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$.

Example 9: Show that the function $\langle \mathbf{u}, \mathbf{v} \rangle = u_1v_1 - u_2v_2 - u_3v_3$ does not define an inner product on R^3, where , $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$.

THEOREM 4.7: PROPERTIES OF INNER PRODUCTS

Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors in an inner product space V, and let c be any real number.

1. $\langle \mathbf{0}, \mathbf{v} \rangle =$ ___________ = ____________

2. $\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle =$ ________________________

Proof:

3. $\langle \mathbf{u}, c\mathbf{v} \rangle =$ ________________________

DEFINITION OF LENGTH, DISTANCE, AND ANGLE

Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in an inner product space V.

1. The length (or _______________) of $\mathbf{u}$ is __________________________.

2. The distance between $\mathbf{u}$ and $\mathbf{v}$ is _________________________________.

3. The angle between and two vectors $\mathbf{u}$ and $\mathbf{v}$ is given by

   ____.

4. $\mathbf{u}$ and $\mathbf{v}$ are orthogonal when _____________________________________.

If _________________, then $\mathbf{u}$ is called a _____________ vector. Moreover, if $\mathbf{v}$ is any nonzero vector in an

inner product space V, then the vector __________________________ is a _______________ vector and is

called the __________________ vector in the ____________________ of $\mathbf{v}$.

Inner product on $C[a,b]$ is $\langle f,g\rangle =$ ____ .

Inner product on $M_{2,2}$ is $\langle A,B\rangle =$ ____ .

Inner product on P_n is $\langle pq\rangle =$ ____ , where

_____________________________________ and ______________________________________.

Example 10: Consider the following inner product defined on R^n:

$\mathbf{u}=(0,-6)$, $\mathbf{v}=(-1,1)$, and $\langle \mathbf{u},\mathbf{v}\rangle = u_1v_1 + 2u_2v_2$

a. Find $\langle \mathbf{u},\mathbf{v}\rangle$

b. Find $\|\mathbf{u}\|$

c. Find $\|\mathbf{v}\|$

d. Find $d(\mathbf{u},\mathbf{v})$

Example 11: Consider the following inner product defined:

$\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)dx$, $f(x) = -x$, $g(x) = x^2 - x + 2$

a. Find $\langle f, g\rangle$

b. Find $\|f\|$

c. Find $\|g\|$

d. Find $d(f,g)$

THEOREM 4.8

Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in an inner product space V.

Cauchy-Schwarz Inequality: ________________________________

Triangle Inequality: ______________________________________

Pythagorean Theorem: $\mathbf{u}$ and $\mathbf{v}$ are orthogonal if and only if

____

Example 12: Verify the triangle inequality for $A = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix}$, and

$\langle A, B \rangle = a_{11}b_{11} + a_{21}b_{21} + a_{12}b_{12} + a_{22}b_{22}$.

DEFINITION OF ORTHOGONAL PROJECTION

Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in an inner product space V, such that $\mathbf{v} \neq \mathbf{0}$. Then the orthogonal projection of $\mathbf{u}$ onto $\mathbf{v}$ is

THEOREM 5.9: ORTHOGONAL PROJECTION AND DISTANCE

Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in an inner product space V, such that $\mathbf{v} \neq \mathbf{0}$. Then

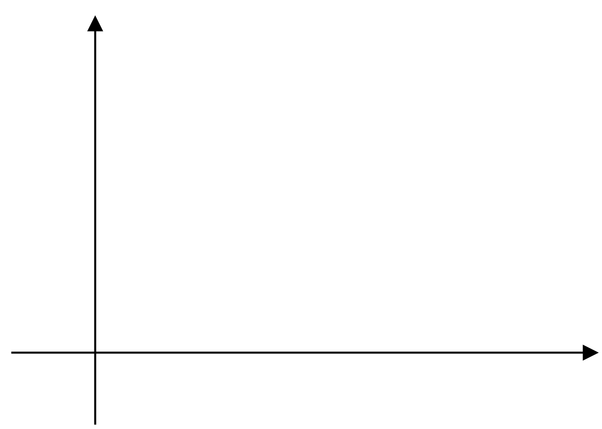
Example 13: Consider the vectors

$\mathbf{u} = (-1, -2)$ and $\mathbf{v} = (4, 2)$. Use the Euclidean inner product to find the following:

a. $\text{proj}_{\mathbf{v}}\mathbf{u}$

b. $\text{proj}_{\mathbf{u}}\mathbf{v}$

c. Sketch the graph of both $\text{proj}_{\mathbf{v}}\mathbf{u}$ and $\text{proj}_{\mathbf{u}}\mathbf{v}$.



4.2: ORTHONORMAL BASES: GRAM-SCHMIDT PROCESS

Learning Objectives:

1. Show that a set of vectors is orthogonal and forms an orthonormal basis, and represent a vector relative to an orthonormal basis
2. Apply the Gram-Schmidt orthonormalization process

Consider the standard basis for R^3, which is

This set is the standard basis because it has useful characteristics such as…The three vectors in the basis are

______________ _________________, and they are each __________________ ______________________.

DEFINITIONS OF ORTHOGONAL AND ORTHONORMAL SETS

A set S of a vector space V is called orthogonal when every pair of vectors in S is orthogonal. If, in addition, each vector in the set is a unit vector, then S is called
____________________________. For $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$, this definition has the following form.

ORTHOGONAL **ORTHONORMAL**

If _____ is a _______________, then it is an ___________________ basis or an ________________________ basis, respectively.

THEOREM 4.10: ORTHOGONAL SETS ARE LINEARLY INDEPENDENT

If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is an orthogonal set of ________________ vectors in an inner product space V, then S is linearly independent.

Proof:

THEOREM 4.10: COROLLARY

If V is an inner product space of dimension n, then any orthogonal set of n nonzero vectors is a basis for V.

Example 1: Consider the following set in R^4.

$$\left\{\left(\frac{\sqrt{10}}{10},0,0,\frac{3\sqrt{10}}{10}\right),(0,0,1,0),(0,1,0,0),\left(-\frac{3\sqrt{10}}{10},0,0,\frac{\sqrt{10}}{10}\right)\right\}$$

a. Determine whether the set of vectors is orthogonal.

b. If the set is orthogonal, then determine whether it is also orthonormal.

c. Determine whether the set is a basis for R^n.

THEOREM 4.11: COORDINATES RELATIVE TO AN ORTHONORMAL BASIS

If $B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is an orthonormal basis for an inner product space V, then the coordinate representation of a vector $\mathbf{w}$ relative to B is

Proof:

The coordinates of ______ relative to the ____________________ basis ____ are called the

____________________ coefficients of ____ relative to ____. The corresponding coordinate matrix of ___

relative to ____ is

Example 2: Show that the set of vectors $\{(2,-5),(10,4)\}$ in R^2 is orthogonal and normalize the set to produce an orthonormal set.

Example 3: Find the coordinate matrix of $\mathbf{x} = (-3, 4)$ relative to the orthonormal basis $B = \left\{ \left(\frac{\sqrt{5}}{5}, \frac{2\sqrt{5}}{5} \right), \left(-\frac{2\sqrt{5}}{5}, \frac{\sqrt{5}}{5} \right) \right\}$ in R^2. Use the dot product as the inner product.

THEOREM 4.12: GRAM-SCHMIDT ORTHONORMALIZATION PROCESS

Let $B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be a basis for an inner product V.

Let $B' = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n\}$, where $\mathbf{w}_i$ is given by

$$\mathbf{w}_1 = \mathbf{v}_1$$

$$\mathbf{w}_2 = \mathbf{v}_2 - \frac{\langle \mathbf{v}_2, \mathbf{w}_1 \rangle}{\langle \mathbf{w}_1, \mathbf{w}_1 \rangle} \mathbf{w}_1$$

$$\mathbf{w}_3 = \mathbf{v}_3 - \frac{\langle \mathbf{v}_3, \mathbf{w}_1 \rangle}{\langle \mathbf{w}_1, \mathbf{w}_1 \rangle} \mathbf{w}_1 - \frac{\langle \mathbf{v}_3, \mathbf{w}_2 \rangle}{\langle \mathbf{w}_2, \mathbf{w}_2 \rangle} \mathbf{w}_2$$

$$\vdots$$

$$\mathbf{w}_n = \mathbf{v}_n - \frac{\langle \mathbf{v}_n, \mathbf{w}_1 \rangle}{\langle \mathbf{w}_1, \mathbf{w}_1 \rangle} \mathbf{w}_1 - \frac{\langle \mathbf{v}_n, \mathbf{w}_2 \rangle}{\langle \mathbf{w}_2, \mathbf{w}_2 \rangle} \mathbf{w}_2 - \cdots - \frac{\langle \mathbf{v}_n, \mathbf{w}_{n-1} \rangle}{\langle \mathbf{w}_{n-1}, \mathbf{w}_{n-1} \rangle} \mathbf{w}_{n-1}$$

Let $\mathbf{u}_i = \frac{\mathbf{w}_i}{\|\mathbf{w}_i\|}$. Then the set $B'' = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ is an orthonormal basis for V. Moreover, $\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\} = \text{span}\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ for $k = 1, 2, \ldots, n$.

Example 4: Apply the Gram-Schmidt orthonormalization process to transform the basis $B=\{(1,0,0),(1,1,1),(1,1,-1)\}$ for a subspace in R^3 into an orthonormal basis. Use the Euclidean inner product on R^3 and use the vectors in the order they are given.

4.3: MATHEMATICAL MODELS AND LEAST SQUARES ANALYSIS

Learning Objectives:

1. When you are done with your homework you should be able to…
2. Define the least squares problem
3. Find the orthogonal complement of a subspace and the projection of a vector onto a subspace
4. Find the four fundamental subspaces of a matrix
5. Solve a least squares problem
6. Use least squares for mathematical modeling

In this section we will study ____________________ systems of linear equations and learn how to find the

________________ ________________ ___________________ of such a system.

LEAST SQUARES PROBLEM

Given an $m \times n$ matrix A and a vector $\mathbf{b}$ in R^m, the _______________ _______________ problem is to

find ____ in R^m such that _________________ is _____________________.

DEFINITION OF ORTHOGONAL SUBSPACES

The subspaces S_1 and S_2 of R^n are orthogonal when _________________ for all $\mathbf{v}_1$ in S_1 and $\mathbf{v}_2$ in S_2.

Example 1: Are the following subspaces orthogonal?

$$S_1 = \text{span}\left\{ \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\} \text{ and } S_2 = \text{span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

DEFINITION OF ORTHOGONAL COMPLEMENT

If S is a subspace of R^n, then the orthogonal complement of S is the set

What's the orthogonal complement of $\{\mathbf{0}\}$ in R^n?

What's the orthogonal complement of R^n?

DEFINITION OF DIRECT SUM

Let S_1 and S_2 be two subspaces of R^n. If each vector _________________ can be uniquely written as the sum of a vector ____ from ____ and a vector ___ from _____, ____________________, then _____ is the direct sum of ____ and ____, and you can write ____________________.

Example 2: Find the orthogonal complement $S^{\perp}$, and find the direct sum $S \oplus S^{\perp}$.

$$S = \text{span}\left\{\begin{bmatrix} 0 \\ 1 \\ -1 \\ 1 \end{bmatrix}\right\}$$

THEOREM 4.13: PROPERTIES OF ORTHOGONAL SUBSPACES

Let S be a subspace of R^n, Then the following properties are true.

1. ______________________________________
2. ______________________________________
3. ______________________________________

THEOREM 4.14: PROJECTION ONTO A SUBSPACE

If $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_t\}$ is an orthonormal basis for the subspace S of R^n, and $\mathbf{v} \in R^n$, then

Example 3: Find the projection of the vector $\mathbf{v}$ onto the subspace S.

$$S = \operatorname{span}\left\{\begin{bmatrix} -1 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}\right\}, \quad \mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

THEOREM 4.15: ORTHOGONAL PROJECTION AND DISTANCE

Let S be a subspace of R^n and let $\mathbf{v} \in R^n$. Then, for all $\mathbf{u} \in S$, $\mathbf{u} \neq \operatorname{proj}_S \mathbf{v}$,

FUNDAMENTAL SUBSPACES OF A MATRIX

Recall that if A is an $m \times n$ matrix, then the column space of A is a ________________ of _____ consisting of all vectors of the form _____, _________. The four fundamental subspaces of the matrix A are defined as follows.

__________ = nullspace of A

__________ = nullspace of A^T

__________ = column space of A

__________ = column space of A^T

Example 4: Find bases for the four fundamental subspaces of the matrix

$$A = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix}.$$

THEOREM 4.16: FUNDAMENTAL SUBSPACES OF A MATRIX

If A is an $m \times n$ matrix, then

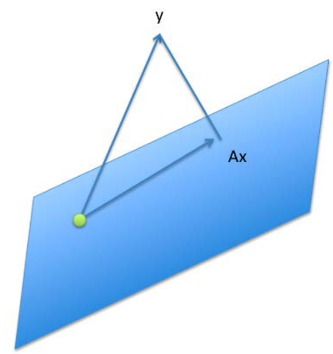
__________ and ___________ are orthogonal subspaces of _____.

__________ and ___________ are orthogonal subspaces of _____.

____

____

SOLVING THE LEAST SQUARES PROBLEM



Recall that we are attempting to find a vector $\mathbf{x}$ that minimizes ______________,

where A is an $m \times n$ matrix and $\mathbf{b}$ is a vector in R^m. Let S be the column space

of A: ________________. Assume that $\mathbf{b}$ is not in S, because otherwise the

system $A\mathbf{x} = \mathbf{b}$ would be _________________________. We are looking for a

vector ______ in ____ that is as close as possible to ___. This desired vector is

the ____________________ of _____ onto ____. So, ___________________

and ________________________ = _______________________ is orthogonal to _________________. However,

this implies that ______________ is in __________, which equals ________________. So, _____________ is in

the __________________ of _____.

The solution of the least squares problem comes down to solving the __________ linear system of equations

_______________________. These equations are called the _________________ equations of the least squares

problem _______________.

Example 5: Find the least squares solution of the system $A\mathbf{x} = \mathbf{b}$.

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}, \ \mathbf{b} = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 2 \end{bmatrix}$$

Example 6: The table shows the numbers of doctoral degrees y (in thousands) awarded in the United States from 2005 through 2008. Find the least squares regression line for the data. Then use the model to predict the number of degrees awarded in 2015. Let t represent the year, with $t = 5$ corresponding to 2005. (Source: U.S. National Center for Education Statistics)

Year	**2005**	**2006**	**2007**	**2008**
Doctoral Degrees, *y*	52.6	56.1	60.6	63.7

4.4: EIGENVALUES AND EIGENVECTORS, AND DIAGONALIZING MATRICES

Learning Objectives:

1. Verify eigenvalues and corresponding eigenvectors
2. Find eigenvectors and corresponding eigenspaces
3. Use the characteristic equation to find eigenvalues and eigenvectors, and find the eigenvalues and eigenvectors of a triangular matrix
4. Find the eigenvalues and eigenvectors of a linear transformation

THE EIGENVALUE PROBLEM

One of the most important problems in linear algebra is the **eigenvalue problem**. When A is an $n \times n$, do nonzero vectors $\mathbf{x}$ in R^n exist such that $A\mathbf{x}$ is a _____________ multiple of $\mathbf{x}$? The scalar, denoted by _____ (____________), is called an ____________________ of the matrix A, and the nonzero vector $\mathbf{x}$ is called an ____________________ of A corresponding to λ.

DEFINITIONS OF EIGENVALUE AND EIGENVECTOR

Let A be an $n \times n$ matrix. The scalar ____ is called an __________________ of A when there is a _____________ vector $\mathbf{x}$ such that _____________. The vector $\mathbf{x}$ is called an __________________ of A corresponding to λ.

*Note that an eigenvector cannot be _____. Why not?

Example 1: Determine whether $\mathbf{x}$ is an eigenvector of A.

$$A = \begin{bmatrix} -3 & 10 \\ 5 & 2 \end{bmatrix}$$

a. $\mathbf{x} = (-8, 4)$

b. $\mathbf{x} = (5, -3)$

THEOREM 4.17: EIGENVECTORS OF λ FORM A SUBSPACE

If A is an $n \times n$ matrix with an eigenvalue λ, then the set of all eigenvectors of λ, together with the zero vector

is a subspace of R^n. This subspace is called the ________________ of λ.

Proof:

THEOREM 4.18: EIGENVALUES AND EIGENVECTORS OF A MATRIX

Let A be an $n \times n$ matrix.

1. An eigenvalue of A is a scalar λ such that _______________.
2. The eigenvectors of A corresponding to λ are the ____________solutions of

   ____________________.

* The equation _________________ is called the _____________________ _____________________ of

A. When expanded to polynomial form, the polynomial is called the ________________

___________________ of A. This definition tells you that the ___________________ of an $n \times n$ matrix

A correspond to the ____________ of the characteristic polynomial of A.

Example 2: Find (a) the characteristic equation and (b) the eigenvalues (and corresponding eigenvectors) of the matrix.

$$A = \begin{bmatrix} 3 & 2 & 1 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix}$$

THEOREM 4.19: EIGENVALUES OF TRIANGULAR MATRICES

If A is an $n \times n$ triangular matrix, then its eigenvalues are the entries on its main ______________________.

Example 3: Find the eigenvalues of the triangular matrix.

$$\begin{bmatrix} -5 & 0 & 0 \\ 3 & 7 & 0 \\ 4 & -2 & 3 \end{bmatrix}$$

EIGENVALUES AND EIGENVECTORS OF LINEAR TRANSFORMATIONS

A number λ is called an ______________________ of a linear transformation ____________ when there is a

________________ vector ____ such that _________________. The vector $\mathbf{x}$ is called an ___________________

of T corresponding to λ, and the set of all eigenvectors of λ (with the zero vector) is called the

__________________ of λ.

Example 4: Consider the linear transformation $T: R^n \to R^n$ whose matrix A relative to the standard base is given. Find (a) the eigenvalues of A, (b) a basis for each of the corresponding eigenspaces, and (c) the matrix A' for T relative to the basis B', where B' is made up of the basis vectors found in part b).

$$A = \begin{bmatrix} -6 & 2 \\ 3 & -1 \end{bmatrix}$$

4.5: DIAGONALIZATION

Learning Objectives:

1. Find the eigenvectors of similar matrices, determine whether a matrix A is diagonalizable, and find a matrix P such that $P^{-1}AP$ is diagonal
2. Find, for a linear transformation $T:V \to V$, a basis B for V such that the matrix for T relative to B is diagonal

DEFINITION OF A DIAGONALIZABLE MATRIX

An $n \times n$ matrix A is diagonalizable when A is similar to a diagonal matrix. That is, A is diagonalizable when there exists an invertible matrix ____ such that ____________ is a diagonal matrix.

THEOREM 4.20: SIMILAR MATRICES HAVE THE SAME EIGENVALUES

If A and B are similar $n \times n$ matrices, then the have the same ___________________________.

Proof:

Example 1: (a) verify that A is diagonalizable by computing $P^{-1}AP$, and (b) use the result of part (a) and Theorem 4.20 to find the eigenvalues of A.

$$A = \begin{bmatrix} 1 & 3 \\ -1 & 5 \end{bmatrix}, \; P = \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}$$

THEOREM 4.21: CONDITION FOR DIAGONALIZATION

An $n \times n$ matrix A is diagonalizable if and only if it has n ________________ ____________________ eigenvectors.

Proof:

Example 2: For the matrix A, find, if possible, a nonsingular matrix P such that $P^{-1}AP$ is diagonal. Verify $P^{-1}AP$ is a diagonal matrix with the eigenvalues on the main diagonal.

$$A = \begin{bmatrix} 4 & 0 & 0 \\ 2 & 2 & 0 \\ 0 & 2 & 2 \end{bmatrix}$$

STEPS FOR DIAGONALIZING AN $n \times n$ SQUARE MATRIX

Let A be an $n \times n$ matrix.

1. Find n linearly independent eigenvectors ________________________ for A (if possible) with corresponding eigenvalues _______________________. If n linearly independent eigenvectors do not exist, then A is not diagonalizable.

2. Let P be the $n \times n$ matrix whose columns consist of these eigenvectors. That is, __________________________. The diagonal matrix ___________________ will have the eigenvalues ______________________ on its main ____________________ (and __________ elsewhere). Note that the order of the eigenvectors used to form P will determine the order in which the eigenvalues appear on the main ___________________ of _____.

THEOREM 4.22: SUFFICIENT CONDITION FOR DIAGONALIZATION

If an $n \times n$ matrix A has ____ ______________ eigenvalues, then the corresponding eigenvectors are ___________________ ___________________ and A is _________________________.

Proof:

Example 3: Find the eigenvalues of the matrix and determine whether there is a sufficient number to guarantee that the matrix is diagonalizable.

$$\begin{bmatrix} 2 & 0 \\ 5 & 2 \end{bmatrix}$$

Example 4: Find a basis B for the domain of T such that the matrix for T relative to B is diagonal.

$$T: R^3 \to R^3 : T(x, y, z) = (-2x + 2y - 3z, 2x + y - 6z, -x - 2y)$$

4.5: SYMMETRIC MATRICES AND ORTHOGONAL DIAGONALIZATION

Learning Objectives:

1. Recognize, and apply properties of, symmetric matrices
2. Recognize, and apply properties of, orthogonal matrices
3. Find an orthogonal matrix P that orthogonally diagonalizes a symmetric matrix A

SYMMETRIC MATRICES

Symmetric matrices arise more often in ________________________ than any other major class of matrices.

The theory depends on both ___________________________ and _____________________________. For

most matrices, you need to go through most of the diagonalization ___________ to ascertain whether a

matrix is _______________________. We learned about one exception, a _____________________ matrix,

which has ____________ entries on the main ________________________. Another type of matrix which

is guaranteed to be ________________________ is a _____________________ matrix.

DEFINITION OF SYMMETRIC MATRIX

A square matrix A is ____________________ when it is equal to its _________________:________________.

Example 1: Determine which of the matrices below are symmetric.

$$A = \begin{bmatrix} -2 & 5 \\ 5 & 1 \end{bmatrix},\ B = \begin{bmatrix} 6 & 5 & 4 \\ 5 & 1 & 0 \\ 4 & 0 & -1 \end{bmatrix},\ C = \begin{bmatrix} 3 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix},\ D = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 7 & 1 & 0 \\ 3 & 1 & 7 & 2 \\ 4 & 0 & 2 & 5 \end{bmatrix}$$

Example 2: Using the diagonalization process, determine if A is diagonalizable. If so, diagonalize the matrix A.

$$A = \begin{bmatrix} 6 & -1 \\ -1 & 5 \end{bmatrix}$$

THEOREM 4.23: PROPERTIES OF SYMMETRIC MATRICES

If A is an $n \times n$ symmetric matrix, then the following properties are true.

1. A is _______________________________.

2. All __________________________ of A are __________.

3. If λ is an ______________________ of A with multiplicity ____, then

   ____ has ____ linearly ___________________ eigenvectors. That is, the

   __________________ of λ has dimension ____.

Proof of Property 1 (for a 2 x 2 symmetric matrix):

Example 3: Prove that the symmetric matrix is diagonalizable.

$$A = \begin{bmatrix} a & a & a \\ a & a & a \\ a & a & a \end{bmatrix}$$

Example 4: Find the eigenvalues of the symmetric matrix. For each eigenvalue, find the dimension of the corresponding eigenspace.

$$A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

DEFINITION OF AN ORTHOGONAL MATRIX

A square matrix P is ___________________ when it is _________________ and when

__________________.

THEOREM 4.24: PROPERTY OF ORTHOGONAL MATRICES

An $n \times n$ matrix P is orthogonal if and only if its ________________________ vectors form an

___________________________ set.

Example 5: Determine whether the matrix is orthogonal. If the matrix is orthogonal, then show that the column vectors of the matrix form an orthonormal set.

$$A = \begin{bmatrix} -\frac{4}{5} & 0 & \frac{3}{5} \\ 0 & 1 & 0 \\ \frac{3}{5} & 0 & \frac{4}{5} \end{bmatrix}$$

THEOREM 4.25: PROPERTY OF SYMMETRIC MATRICES

Let A be an $n \times n$ symmetric matrix. If λ_1 and λ_2 are ________________ eigenvalues of A, then their corresponding ____________________ $\mathbf{x}_1$ and $\mathbf{x}_2$ are _____________________.

THEOREM 4.26: FUNDAMENTAL THEOREM OF SYMMETRIC MATRICES

Let A be an $n \times n$ matrix. Then A is ____________________________ _______________________ and has ____________ eigenvalues if and only if A is _________________________.

Proof:

STEPS FOR DIAGONALIZING A SYMMETRIC MATRIX

Let A be an $n \times n$ symmetric matrix.

1. Find all ______________________ of A and determine the ______________________ of each.
2. For ____________ eigenvalue of multiplicity _____, find a _________eigenvector. That is, find any _____________________ and then __________________ it.
3. For ____________ eigenvalue of multiplicity ________, find a set of _____ ____________________ __________________________ eigenvectors. If this set is not _______________________, apply the ________________________ ____________________________ process.
4. The results of steps 2 and 3 produce an ___________________ set of _______ eigenvectors. Use these eigenvectors to form the _________ of ______. The matrix ___________________________ will be ___________________. The main entries of _____ are the ____________________ of _____.

Example 5: Find a matrix P such that $P^T AP$ orthogonally diagonalizes A. Verify that $P^T AP$ gives the proper diagonal form.

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

Example 6: Prove that if a symmetric matrix A has only one eigenvalue λ, then $A = \lambda I$.

4.6: APPLICATIONS OF EIGENVALUES AND EIGENVECTORS

Learning Objectives:

1. Find the matrix of a quadratic form and use the Principal Axes Theorem to perform a rotation of axes for a conic and a quadric

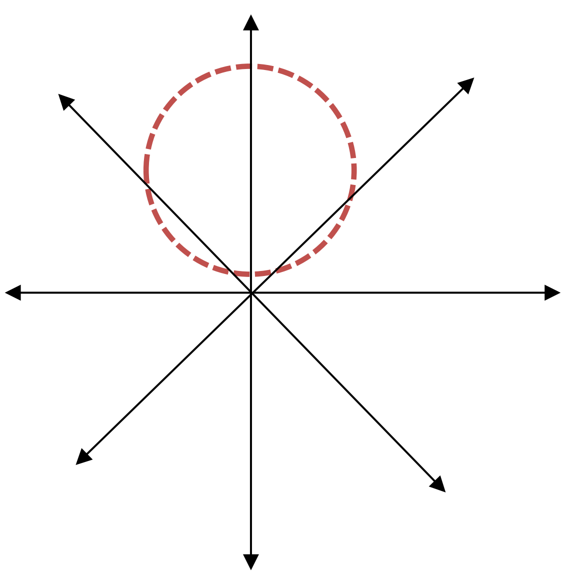
QUADRATIC FORMS

Every conic section in the *xy*-plane can be written as:

If the equation of the conic has no *xy*-term (______________), then the axes of the graphs are parallel to the coordinate axes. For second-degree equations that have an *xy*-term, it is helpful to first perform a

___________________ of axes that eliminates the *xy*-term. The required rotation angle is $\cot 2\theta = \frac{a-c}{b}$. With this rotation, the standard basis for R^2, _____________________ is rotated to form the new basis

_____________________________________.

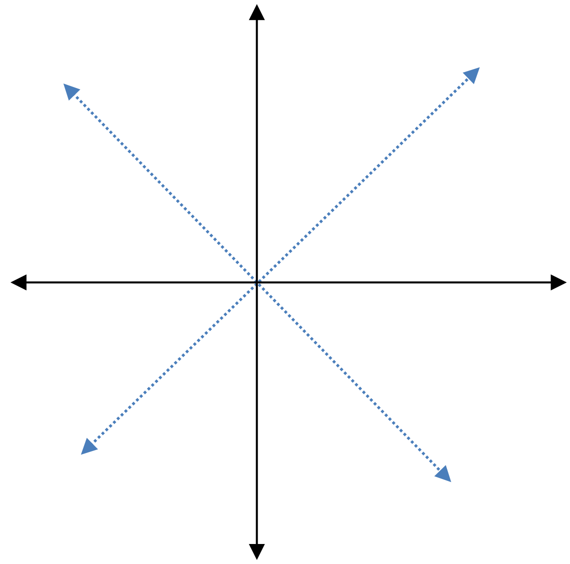


Example 1: Find the coordinates of a point (x, y) in R^2 relative to the basis $B' = \{(\cos\theta, \sin\theta), (-\sin\theta, \cos\theta)\}$.

ROTATION OF AXES

The general second-degree equation $ax^2 + bxy + cy^2 + dx + ey + f = 0$ can be written in the form $a'(x')^2 + c'(y')^2 + d'x' + e'y' + f' = 0$ by rotating the coordinate axes counterclockwise through the angle θ, where θ is defined by $\cot 2\theta = \frac{a-c}{b}$. The coefficients of the new equation are obtained from the substitutions $x = x'\cos\theta - y'\sin\theta$ and $y = x'\sin\theta + y'\cos\theta$.

Example 2: Perform a rotation of axes to eliminate the *xy*-terms in $5x^2 - 6xy + 5y^2 + 14\sqrt{2}x - 2\sqrt{2}y + 18 = 0$. Sketch the graph of the resulting equation.



________________________ and _________________________ can be used to solve the rotation of axes

problem. It turns out that the coefficients *a'* and *c'* are eigenvalues of the matrix

The expression _______________________ is called the _________________ form associated with the

quadratic equation and the matrix ____ is called the _______________ of the _______________

form. Note that ____ is ____________________. Moreover, ___ will be ________________________ if and

only if its corresponding quadratic form has no ______ term.

Example 3: Find the matrix of quadratic form associated with each quadratic equation.

a. $x^2 + 4y^2 + 4 = 0$

b. $5x^2 - 6xy + 5y^2 + 14\sqrt{2}x - 2\sqrt{2}y + 18 = 0$

Now, let's check out how to use the matrix of quadratic form to perform a rotation of axes.

Let $X = \begin{bmatrix} x \\ y \end{bmatrix}$. Then the quadratic expression $ax^2 + bxy + cy^2 + dx + ey + f$ can be written in matrix form as follows:

If ______________, then no ____________________ is necessary. But if _____________, then because ___ is

symmetric, you may conclude that there exists an ___________________ matrix ____ such that ___________

is diagonal. So, if you let

then it follows that __________, and

The choice of ___ must be made with care. Since ___ is orthogonal, its determinant will be __________. If P is chosen so that $|P| = 1$, then P will be of the form

where θ gives the angle of rotation of the conic measured from the __________ *x*-axis to the positive *x'*-axis.

PRINCIPAL AXES THEOREM

For a conic whose equation is $ax^2 + bxy + cy^2 + dx + ey + f = 0$, the rotation given by __________________ eliminates the *xy*-term when P is an orthogonal matrix, with $|P| = 1$, that diagonalizes A. That is

where λ_1 and λ_2 are eigenvalues of A. The equation of the rotated conic is given by

Example 4: Use the Principal Axes Theorem to perform a rotation of axes to eliminate the *xy*-term in the quadratic equation. Identify the resulting rotated conic and give its equation in the new coordinate system.

$5x^2 - 6xy + 5y^2 + 14\sqrt{2}x - 2\sqrt{2}y + 18 = 0$